Theoretical status of the muon $g-2$

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SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

Basics of the anomalous magnetic moment

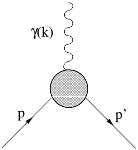
Electrostatic properties of charged particles:
Charge Q, Magnetic moment $\vec{\mu}$, Electric dipole moment $\vec{d}$

For a spin 1/2 particle:

$$\vec{\mu} = g\frac{e}{2m}\vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1+a), \qquad a = \frac{1}{2}(g-2) : \text{anomalous magnetic moment}$$

Long interplay between experiment and theory: structure of fundamental forces

In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{\mathrm{u}}(p')\left[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}}\right]\mathrm{u}(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

a_e: Test of QED. Most precise determination of $\alpha = e^2/4\pi$.
a_μ: Less precisely measured than a_e, but all sectors of Standard Model (SM), i.e. QED, Weak and QCD (hadronic), contribute significantly.
Sensitive to possible contributions from New Physics. Often (but not always !):

$$a_\ell \sim \left(\frac{m_\ell}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 43000 \text{ more sensitive than } a_e \text{ [exp. precision} \to \text{factor 19]}$$

Muon $g-2$: current status

Contribution	$a_\mu \times 10^{11}$	Reference
QED (leptons)	$116\,584\,718.853 \pm 0.036$	Aoyama et al. '12
Electroweak	153.6 ± 1.0	Gnendiger et al. '13
HVP: LO	6907.5 ± 47.2	Jegerlehner, Szafron '11
NLO	-100.3 ± 2.2	Jegerlehner, Szafron '11
NNLO	12.4 ± 0.1	Kurz et al. '14
HLbL	116 ± 40	Jegerlehner, AN '09
NLO	3 ± 2	Colangelo et al. '14
Theory (SM)	$116\,591\,811 \pm 62$	
Experiment	$116\,592\,089 \pm 63$	Bennett et al. (BNL) '06
Experiment - Theory	278 ± 88	3.1 σ

HVP: Hadronic vacuum polarization

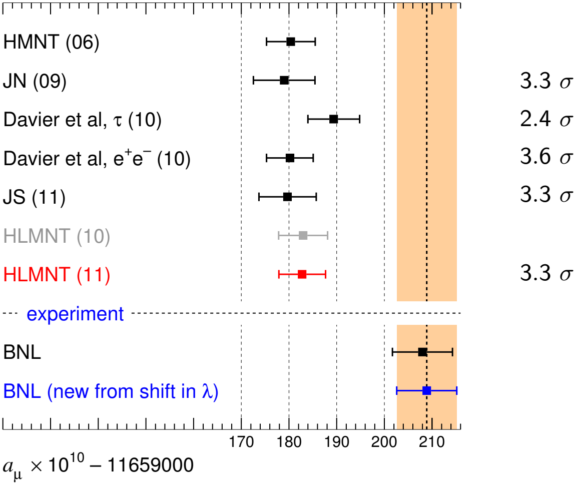
HLbL: Hadronic light-by-light scattering

Other estimate: $a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09).

Discrepancy a sign of New Physics ? General MSSM with light sleptons, sneutrinos, charginos, neutralinos ($M_{\text{SUSY}} \sim 100 - 300$ GeV) and large $\tan\beta \sim 10 - 30$:
$a_\mu^{\text{SUSY}} \approx 130 \times 10^{-11} \, (100 \text{ GeV}/M_{\text{SUSY}})^2 \tan\beta$ (Czarnecki, Marciano '01)

Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab and JPARC with $\delta a_\mu = 16 \times 10^{-11}$. Way forward for HVP clear: more precise measurements of $\sigma(e^+e^- \to \text{hadrons})$.
Not so obvious how to improve HLbL !

Muon $g-2$: other recent evaluations



Source: Hagiwara et al. '11. Note units of 10^{-10} !

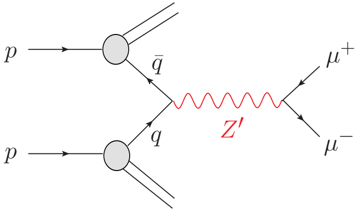
Aoyama et al. '12: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (249 \pm 87) \times 10^{-11}$ [2.9 σ]

Benayoun et al. '13: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 394 \times 10^{-11}$ [4.9 σ]

Tests of the Standard Model and search for New Physics

Search for New Physics with two complementary approaches:

1. High Energy Physics:
   e.g. Large Hadron Collider (LHC) at CERN
   Direct production of new particles
   e.g. heavy Z' $\Rightarrow$ resonance peak in invariant mass distribution of $\mu^+\mu^-$ at $M_{Z'}$.

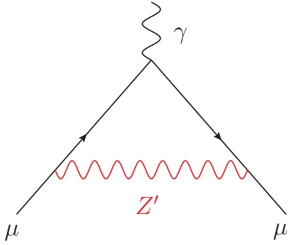


2. Precision physics:
   e.g. anomalous magnetic moments a_e, a_μ
   Indirect effects of virtual particles in quantum corrections
   $\Rightarrow$ Deviations from precise predictions in SM

$$\text{For } M_{Z'} \gg m_\ell : \quad a_\ell \sim \left(\frac{m_\ell}{M_{Z'}}\right)^2$$

Note: there are also non-decoupling contributions of heavy New Physics !
Another example: new light vector meson ("dark photon") with $M_V \sim (10 - 100)$ MeV.

a_e, a_μ allow to exclude some models of New Physics or to constrain their parameter space.



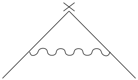
Some theoretical comments

- Anomalous magnetic moment is finite and calculable
  Corresponds to effective interaction Lagrangian of mass dimension 5:

$$\mathcal{L}_{\text{eff}}^{\text{AMM}} = -\frac{e_\ell a_\ell}{4m_\ell} \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$

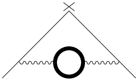
  $a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet divergence.

- Anomalous magnetic moments are dimensionless
  To lowest order in perturbation theory in quantum electrodynamics (QED):



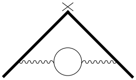
$$= a_e = a_\mu = \frac{\alpha}{2\pi} \qquad \text{[Schwinger '47/'48]}$$

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple (not always !):



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Electron $g-2$

Main contribution in Standard Model from mass-independent Feynman diagrams in QED with electrons in internal lines (perturbative series in α):

$$\begin{aligned} a_e^{\mathrm{SM}} \quad = \quad & \sum_{n=1}^{5} c_n \left(\frac{\alpha}{\pi}\right)^n \\ & +2.7478(2) \times 10^{-12} \quad [\text{Loops in QED with } \mu, \tau] \\ & +0.0297(5) \times 10^{-12} \quad [\text{weak interactions}] \\ & +1.682(20) \times 10^{-12} \quad [\text{strong interactions / hadrons}] \end{aligned}$$

The numbers are based on the paper by Aoyama et al. '12.

With the known coefficients c_n up to 5-loops in QED and the experimental value $a_e^{\exp} = (1\ 159\ 652\ 180.73 \pm 0.28) \times 10^{-12}$ [0.24ppb] from Hanneke et al. '08, one obtains the most precise determination of the fine structure constant by assuming $a_e^{\exp} = a_e^{\mathrm{SM}}$ and solving for α:

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1657\ \underbrace{(68)}_{c_4}\ \underbrace{(46)}_{c_5}\ \underbrace{(24)}_{\text{had+EW}}\ \underbrace{(331)}_{a_e^{\exp}}\ [342] \quad [0.25\text{ppb}]$$

QED: mass-independent contributions to a_e

- α: 1-loop, 1 Feynman diagram; Schwinger '47/'48:

  $c_1 = \frac{1}{2}$

- α^2: 2-loops, 7 Feynman diagrams; Petermann '57, Sommerfield '57:

  $c_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2}\ln 2 + \frac{3}{4}\zeta(3) = -0.32847896557919378\ldots$

- α^3: 3-loops, 72 Feynman diagrams; ..., Laporta, Remiddi '96:

$$\begin{aligned} c_3 &= \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2\ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 \\ &\quad + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2\ln^2 2\right\} \\ &= 1.181241456587\ldots \end{aligned}$$

- α^4: 4-loops, 891 Feynman diagrams; Kinoshita et al. '99, ..., Aoyama et al. '08; '12:

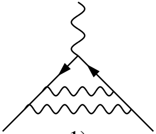
  $c_4 = -1.9106(20)$ (numerical evaluation)

- α^5: 5-loops, 12672 Feynman diagrams; Aoyama et al. '05, ..., '12:

  $c_5 = 9.16(58)$ (numerical evaluation)

  Replaces earlier rough estimate $c_5 = 0.0 \pm 4.6$.
  Result removes biggest theoretical uncertainty in a_e !

Mass-independent 2-loop Feynman diagrams in a_e



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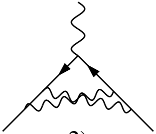

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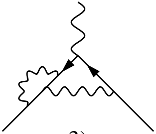

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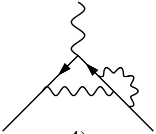

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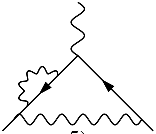

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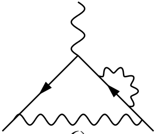

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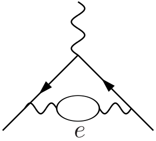
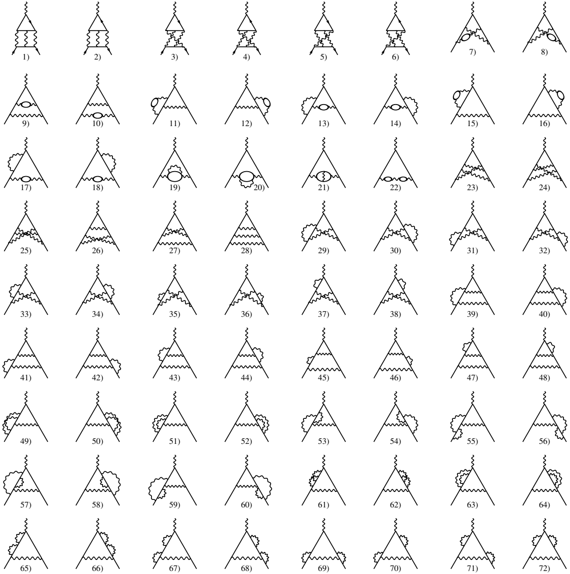


7)

Mass-independent 3-loop Feynman diagrams in a_e



Muon $g-2$

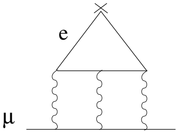
In Standard Model (SM):

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$$

In contrast to a_e, here now the contributions from weak and strong interactions (hadrons) are relevant, since $a_\mu \sim (m_\mu/M)^2$.

QED contributions

- Diagrams with internal electron loops are enhanced.
- At 2-loops: vacuum polarization from electron loops enhanced by QED short-distance logarithm
- At 3-loops: light-by-light scattering from electron loops enhanced by QED infrared logarithm [Aldins et al. '69, '70; Laporta, Remiddi '93]



$+ \ldots$

$$a_\mu^{(3)}\Big|_{\text{lbyl}} = \left[\frac{2}{3}\pi^2 \ln \frac{m_\mu}{m_e} + \ldots\right]\left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots\left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's suppressed (decoupling)

QED result up to 5 loops

Include contributions from all leptons (Aoyama et al. '12):

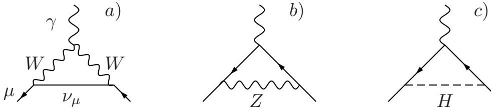
$$
\begin{aligned}
a_\mu^{\text{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$

- Earlier evaluation of 5-loop contribution yielded $c_5 = 662(20)$ (Kinoshita, Nio '06, numerical evaluation of 2958 diagrams, known or likely to be enhanced). New value is 4.5σ from this leading log estimate and 20 times more precise.
- Aoyama et al. '12: What about the 6-loop term ? Leading contribution from light-by-light scattering with electron loop and insertions of vacuum-polarization loops of electrons into each photon line $\Rightarrow a_\mu^{\text{QED}}(\text{6-loops}) \sim 0.1 \times 10^{-11}$

Contributions from weak interaction

Numbers from recent reanalysis by Gnendiger et al. '13.

1-loop contributions [Jackiw + Weinberg, 1972; ...]:



$$a_\mu^{\text{weak, (1)}}(W) = \frac{\sqrt{2} G_\mu m_\mu^2}{16\pi^2} \frac{10}{3} + \mathcal{O}(m_\mu^2/M_W^2) = 388.70 \times 10^{-11}$$

$$a_\mu^{\text{weak, (1)}}(Z) = \frac{\sqrt{2} G_\mu m_\mu^2}{16\pi^2} \frac{(-1+4s_W^2)^2 - 5}{3} + \mathcal{O}(m_\mu^2/M_Z^2) = -193.89 \times 10^{-11}$$

Contribution from Higgs negligible: $a_\mu^{\text{weak, (1)}}(H) \leq 5 \times 10^{-14}$ for $m_H \geq 114$ GeV.

$$a_\mu^{\text{weak, (1)}} = (194.80 \pm 0.01) \times 10^{-11}$$

2-loop contributions (1678 diagrams) [Czarnecki et al. '95, '96; ...]:

$$a_\mu^{\text{weak, (2)}} = (-41.2 \pm 1.0) \times 10^{-11}, \quad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$$

Total weak contribution:

$$a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$$

Under control ! With knowledge of $M_H = 125.6 \pm 1.5$ GeV, uncertainty now mostly hadronic $\pm 1.0 \times 10^{-11}$ (Peris et al. '95; Knecht et al. '02; Czarnecki et al. '03, '06). 3-loop effects via RG: $\pm 0.20 \times 10^{-11}$ (Degrassi, Giudice '98; Czarnecki et al. '03).

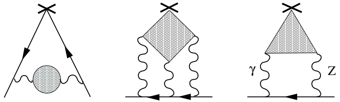
Hadronic contributions to the muon $g-2$: largest source of error

- QCD: quarks bound by strong gluonic interactions into hadronic states
- In particular for the light quarks u, d, s $\rightarrow$ cannot use perturbation theory !

Possible approaches to QCD at low energies:

- Lattice QCD: often still limited precision
- Effective quantum field theories with hadrons (ChPT): limited validity
- Simplifying hadronic models: model uncertainties not controllable
- Dispersion relations: extend validity of EFT's, reduce model dependence, often not all the needed input data available

Different types of contributions to $g-2$:



(a) (b) (c)

Light quark loop not well defined
$\rightarrow$ Hadronic "blob"

(a) Hadronic vacuum polarization (HVP) $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

(b) Hadronic light-by-light scattering (HLbL) $\mathcal{O}(\alpha^3), \mathcal{O}(\alpha^4)$

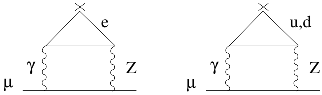
(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

2-Loop EW

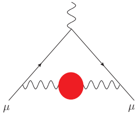
Small hadronic uncertainty from triangle diagrams.

Anomaly cancellation within each generation !

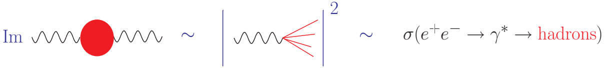
Cannot separate leptons and quarks !



Hadronic vacuum polarization

$a_\mu^{\mathrm{HVP}} =$ 

Optical theorem (from unitarity; conservation of probability) for hadronic contribution $\to$ dispersion relation:

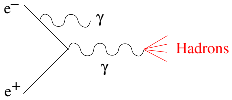


$$a_\mu^{\mathrm{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

[Bouchiat, Michel ’61; Durand ’62; Brodsky, de Rafael ’68; Gourdin, de Rafael ’69]

$K(s)$ slowly varying, positive function $\Rightarrow$ a_μ^{HVP} positive. Data for hadronic cross section σ at low center-of-mass energies $\sqrt{s}$ important due to factor $1/s$: $\sim 70\%$ from $\pi\pi$ [$\rho(770)$] channel, $\sim 90\%$ from energy region below 1.8 GeV.

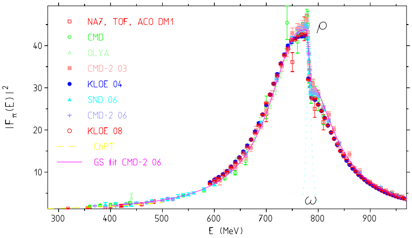
Other method instead of energy scan: “Radiative return” at colliders with fixed center-of-mass energy (DAΦNE, B-Factories, BEPC) [Binner et al. ’99; Czyż et al. ’00-’03]



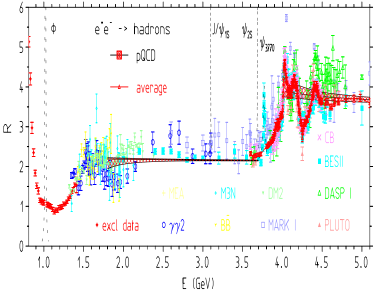
Measured hadronic cross-section

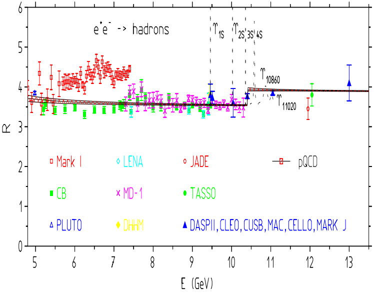
Pion form factor $|F_\pi(E)|^2$
($\pi\pi$-channel)

$$R(s) \quad = \quad \frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$$
$$(4m_\pi^2 < s < 9m_\pi^2)$$



R-ratio:





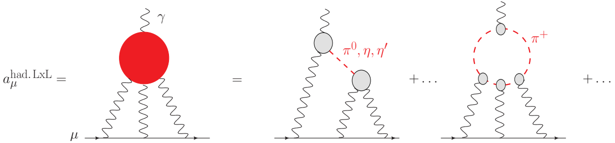
Jegerlehner, AN '09

Hadronic vacuum polarization: some recent evaluations

Authors	Contribution to $a_\mu^{\mathrm{HVP}} \times 10^{11}$
Jegerlehner '08; JN '09 (e^+e^-)	6903.0 ± 52.6
Davier et al. '09 (e^+e^-) [$+\ \tau$]	6955 ± 41 [7053 ± 45]
Teubner et al. '09 (e^+e^-)	6894 ± 40
Davier et al. '11 (e^+e^-) [$+\ \tau$]	6923 ± 42 [7015 ± 47]
Jegerlehner, Szafron '11 (e^+e^-) [$+\ \tau$]	6907.5 ± 47.2 [6909.6 ± 46.5]
Hagiwara et al. '11 (e^+e^-)	6949.1 ± 42.7
Benayoun at al. '12 ($e^+e^- + \tau$: HLS improved)	6877.2 ± 46.3

- Precision: $< 1\%$. Non-trivial because of radiative corrections (radiated photons).
- Even if values for a_μ^{HVP} after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, BES III, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding.
- Use of τ data: additional sources of isospin violation ? Ghozzi, Jegerlehner '04; Benayoun et al. '08, '09; Wolfe, Maltman '09; Jegerlehner, Szafron '11 ($\rho - \gamma$-mixing), also included in HLS-approach by Benayoun et al. '12.
- Lattice QCD: Various groups are working on it, precision at level of 5-10%, not yet competitive with phenomenological evaluations.

Hadronic light-by-light scattering (HLbL) in $g-2$



- Only model calculations so far: large uncertainties, difficult to control.
- Frequently used estimates:

$$a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11} \qquad \text{(Prades, de Rafael, Vainshtein '09)}$$
$$a_\mu^{\text{HLbL}} = (116 \pm 40) \times 10^{-11} \qquad \text{(AN '09; Jegerlehner, AN '09)}$$

  Based almost on same input: calculations by various groups using different models for individual contributions. Error estimates are mostly guesses !
- Need much better understanding of complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$ ($\delta a_\mu(\text{future exp}) = 16 \times 10^{-11}$).
- Recent new proposal: Colangelo et al. '14, '15; Pauk, Vanderhaeghen '14: use dispersion relations (DR) to connect contribution to HLbL from light pseudoscalars to in principle measurable form factors and cross-sections:

$$\gamma^*\gamma^* \rightarrow \pi^0, \eta, \eta'$$
$$\gamma^*\gamma^* \rightarrow \pi\pi$$

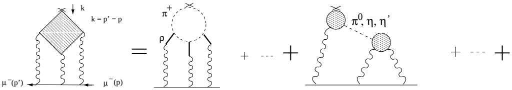
  Could connect HLbL uncertainty to exp. measurement errors, like HVP.
- Maybe in future: HLbL from Lattice QCD. First steps: Blum et al. '05, . . . , '14. New approach: Mainz group (Asmussen et al.).

Current approach to HLbL scattering

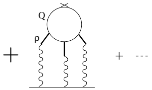
Use hadronic model at low energies with exchanges and loops of resonances and some (dressed) "quark-loop" at high energies.

Classification of de Rafael '94

Chiral counting p^2 (from Chiral Perturbation Theory (ChPT)) and large-N_C counting as guideline to classify contributions (all higher orders in p^2 and N_C contribute):



\+ ··· + Exchange of other resonances $(f_0, a_1, f_2 \ldots)$ +



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

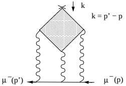
Relevant scales in HLbL ($\langle VVVV \rangle$ with off-shell photons !): $0 - 2$ GeV $\gg m_\mu$!

Constrain models using experimental data (processes of hadrons with photons: decays, form factors, scattering) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

Problem: four-point function depends on several invariant momenta $\Rightarrow$ distinction between low and high energies not as easy as for two-point function in HVP.
Mixed regions, where one loop momentum Q_1^2 is large and the other Q_2^2 is small and vice versa.

General analysis of four-point function $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3)$ relevant for $g - 2$: Bijnens et al. '96; Bijnens (Talk at $g - 2$ Workshop, Mainz, '14); Eichmann et al. '14, '15; Colangelo et al. '15.

HLbL scattering: Summary of selected results



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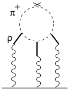

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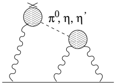
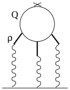


+ ··· + Exchange of other resonances $(f_0, a_1, f_2 \ldots)$ +



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Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	+2.3 [c-quark]
N,JN:	+116 (40)	-19 (13)	+99 (16)	+15 (7) $[f_0, a_1]$	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Pseudoscalars: numerically dominant contribution (according to most models !).

Recall (in units of 10^{-11}): $\delta a_\mu(\text{HVP}) \approx 45$; $\delta a_\mu(\text{exp [BNL]}) = 63$; $\delta a_\mu(\text{future exp}) = 16$

Recent developments

- Most calculations for neutral pion and all light pseudoscalars agree at level of 15%, but some are quite different:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL;PS}} = (59 - 114) \times 10^{-11}$$

- New estimates for axial vectors (Pauk, Vanderhaeghen '14; Jegerlehner '14):

$$a_\mu^{\mathrm{HLbL;axial}} = (6 - 8) \times 10^{-11}$$

  Substantially smaller than in MV '04 !

- First estimate for tensor mesons (Pauk, Vanderhaeghen '14):

$$a_\mu^{\mathrm{HLbL;tensor}} = 1 \times 10^{-11}$$

- Open problem: Dressed pion-loop
  Potentially important effect from pion polarizability and a_1 resonance (Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = -(11 - 71) \times 10^{-11}$$

  Maybe large negative contribution, in contrast to BPP '96, HKS '96.

- Open problem: Dressed quark-loop
  Dyson-Schwinger equation approach (Fischer, Goecke, Williams '11, '13):

$$a_\mu^{\mathrm{HLbL;quark-loop}} = 107 \times 10^{-11} \qquad \text{(still incomplete !)}$$

  Large contribution, no damping seen, in contrast to BPP '96, HKS '96.

Data driven approach to HLbL using dispersion relations (DR)

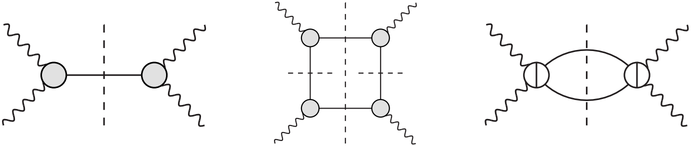
Strategy: Split contributions to HLbL into two parts:

I: Data-driven evaluation using DR (hopefully numerically dominant):
  (1) π^0, η, η' poles
  (2) $\pi\pi$ intermediate state

II: Model dependent evaluation (hopefully numerically subdominant):
  (1) Axial vectors (3π-intermediate state), ...
  (2) Quark-loop, matching with pQCD

Error goals: Part I: 10% precision (data driven), Part II: 30% precision.
To achieve overall error of about 20% ($\delta a_\mu^{\rm HLbL} = 20 \times 10^{-11}$).

Colangelo et al.:
Classify intermediate states in four-point function, then project onto $g-2$.



$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0}_{\mu\nu\lambda\sigma} + \Pi^{\rm FsQED}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \cdots$

$\Pi^{\pi^0}_{\mu\nu\lambda\sigma}$ = pion pole (similarly for η, η')
$\Pi^{\rm FsQED}_{\mu\nu\lambda\sigma}$ = scalar QED with vertices dressed by pion vector form factor F_π^V
$\Pi^{\pi\pi}_{\mu\nu\lambda\sigma}$ = remaining $\pi\pi$ contribution

Pauk, Vanderhaeghen: Write DR directly for Pauli form factor $F_2(k^2)$.

DR approach to HLbL: $\pi\pi$ intermediate state

Colangelo et al., arXiv:1402.7081, arXiv:1408.2517

The remaining $\pi\pi$ contribution $\Pi^{\pi\pi}_{\mu\nu\lambda\sigma}$ is given by

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i I_i\left(s, q_1^2, q_2^2\right) T_i^{\pi\pi}\left(q_1, q_2; p\right)}{q_1^2 q_2^2 s Z_1 Z_2}$$

with $Z_1 = (p+q_1)^2 - m^2$, $Z_2 = (p-q_2)^2 - m^2$, $s = (q_1+q_2)^2$, $p^2 = m^2$ and known kinematical functions $T_i^{\pi\pi}(q_1, q_2; p)$, while information on the scattering amplitude on the cut is given by the dispersive integrals $I_i(s, q_1^2, q_2^2)$.

For first S-wave:

$$I_1\left(s, q_1^2, q_2^2\right) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'-s} \left[\left(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda\left(s', q_1^2, q_2^2\right)} \right) \mathrm{Im}\, h^0_{++,++}\left(s'; q_1^2, q_2^2; s, 0\right) \right.$$
$$\left. + \frac{2\xi_1\xi_2}{\lambda\left(s', q_1^2, q_2^2\right)} \mathrm{Im}\, h^0_{00,++}\left(s'; q_1^2, q_2^2; s, 0\right) \right]$$

ξ_i: normalization of longitudinal polarization vectors of off-shell photons

$h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s; q_1^2, q_2^2; q_3^2, q_4^2)$: partial-wave helicity amplitudes with angular momentum J for process $\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \gamma^*(q_3,\lambda_3)\gamma^*(q_4,\lambda_4)$.

Partial-wave unitarity relates imaginary parts in integrals I_i to helicity partial waves $h_{J,\lambda_1\lambda_2}(s; q_1^2, q_2^2)$ for $\gamma^*\gamma^* \to \pi\pi$, which have to be determined from experiment:

$$\mathrm{Im}\, h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}\left(s; q_1^2, q_2^2; q_3^2, q_4^2\right) = \frac{\sqrt{1-4M_\pi^2/s}}{16\pi} h_{J,\lambda_1\lambda_2}\left(s; q_1^2, q_2^2\right) h_{J,\lambda_3\lambda_4}\left(s; q_3^2, q_4^2\right)$$

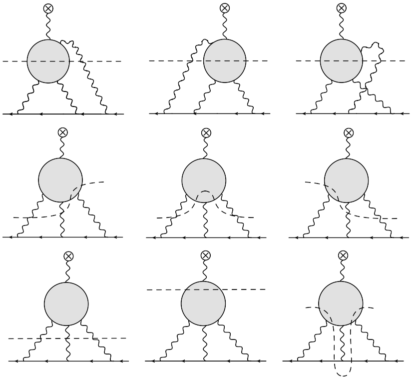
Another approach to HLbL using DR's

Pauk, Vanderhaeghen, arXiv:1403.7503, arXiv:1409.0819

Write DR directly for form factor $F_2(k^2)$:

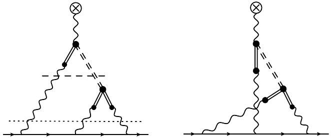
$$F_2(0) = \frac{1}{2\pi i} \int_0^\infty \frac{\mathrm{d}k^2}{k^2} \mathrm{Disc}_{k^2} F_2(k^2)$$

Then study contributions from different intermediate states from the cuts in the unitarity diagrams related to measurable physical processes like $\gamma^*\gamma^* \to X$ and $\gamma^* \to \gamma X$:



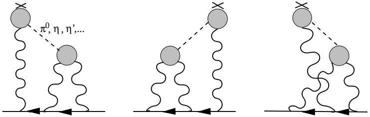
Pseudoscalar pole contribution

Considering the pseudoscalar intermediate state and VMD form factor ($\rho - \gamma$ mixing) it was shown that from dispersion relation one obtains precisely the result given by direct evaluation of two-loop integral in Knecht, AN '02 for the pseudoscalar pole.



Dashed: two-particle cut
Dotted: three-particle cut
Double-dashed: pseudoscalar pole
Double-solid: vector meson pole

Pion-pole contribution



Knecht, Nyffeler '02:

$$
\begin{aligned}
a_\mu^{\mathrm{HLbL};\pi^0} &= \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right] \\
a_\mu^{\mathrm{HLbL};\pi^0(1)} &= \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\quad \times \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2, 0)}{q_2^2 - m_\pi^2}\ \tilde{T}_1(q_1, q_2; p) \\
a_\mu^{\mathrm{HLbL};\pi^0(2)} &= \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\quad \times \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\ \mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2}\ \tilde{T}_2(q_1, q_2; p)
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Pion-pole contribution determined by measurable pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ (on-shell pion, one or two off-shell photons). Currently, only single-virtual TFF $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-q^2, 0)$ has been measured by CELLO, CLEO, BABAR, Belle for (mostly) spacelike momenta. Analysis ongoing at BES III, measurement planned at KLOE-2. Measurement of double-virtual form factor planned at BES III. Analogously for η, η'-pole contributions.

3-dimensional integral representation

Jegerlehner, AN '09:

$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right]$$

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 dt\, \tilde{I}_1(Q_1,Q_2,t)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2,(Q_1+Q_2)^2)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_2^2,0)$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 dt\, \tilde{I}_2(Q_1,Q_2,t)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}((Q_1+Q_2)^2,0)$$

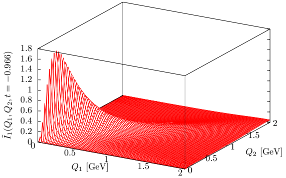
$$\tilde{I}_1(Q_1,Q_2,t) = \left(-\frac{2\pi}{3}\right)\sqrt{1-t^2}\,\frac{Q_1^3 Q_2^3}{Q_2^2+m_\pi^2}\, I_1(Q_1,Q_2,t)$$

$$\tilde{I}_2(Q_1,Q_2,t) = \left(-\frac{2\pi}{3}\right)\sqrt{1-t^2}\,\frac{Q_1^3 Q_2^3}{(Q_1+Q_2)^2+m_\pi^2}\, I_2(Q_1,Q_2,t)$$

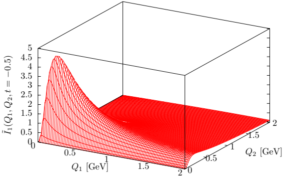
- After Wick rotation: Q_1, Q_2 are Euclidean (spacelike) four-momenta. Integrals run over the lengths of the four-vectors with $Q_i \equiv |(Q_i)_\mu|, i = 1,2$ and angle θ between them: $Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta,\ t = \cos\theta$.
- Separation of generic kinematics described by model-independent weight functions $\tilde{I}_{1,2}(Q_1,Q_2,t)$ and double-virtual form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)$ which can in principle be measured or obtained from DR for form factor itself. Formally: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$.
- $\tilde{I}_{1,2}(Q_1,Q_2,t)$: dimensionless. $\tilde{I}_2(Q_1,Q_2,t)$ symmetric under $Q_1 \leftrightarrow Q_2$.
- $\tilde{I}_{1,2}(Q_1,Q_2,t) \to 0$ for $Q_{1,2} \to 0$. $\tilde{I}_{1,2}(Q_1,Q_2,t) \to 0$ for $t \to \pm 1$.

Relevant momentum regions in $a_\mu^{\mathrm{HLbL};\pi^0}$ (work in preparation)

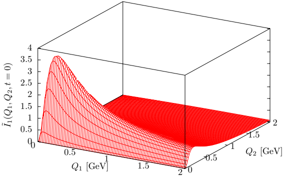
Weight function $\tilde{I}_1(Q_1, Q_2, t)$:



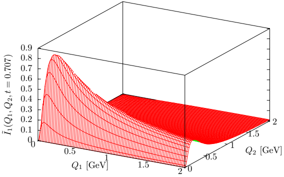
$t = -0.966, \quad \theta = 165^\circ$



$t = -0.5, \quad \theta = 120^\circ$



$t = 0, \quad \theta = 90^\circ$



$t = 0.707, \quad \theta = 45^\circ$

Low momentum region most important. Peak around $Q_1 \sim 0.2$ GeV, $Q_2 \sim 0.15$ GeV.
For $t > -0.85$ ($\theta < 150^\circ$) a ridge develops along Q_1 direction for $Q_2 \sim 0.2$ GeV.
Leads for constant form factor to a divergence $\ln^2 \Lambda$ for some momentum cutoff Λ.
Realistic form factor falls off for large Q_i and integral $a_\mu^{\mathrm{HLbL};\pi^0(1)}$ will be convergent.

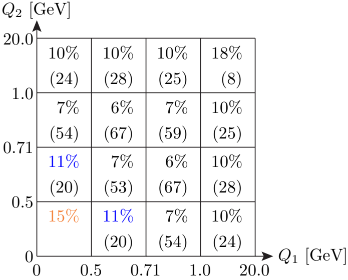
$\tilde{I}_2$: about a factor 10 smaller than $\tilde{I}_1$ and there is no ridge. Peak for $Q_1 \sim 0.15$ GeV, $Q_2 \sim 0.15$ GeV for t near -1, peak moves to lower Q_i values when t grows.
Even for constant form factor, one obtains finite result: $\left(\frac{\alpha}{\pi}\right)^3 a_\mu^{\mathrm{HLbL};\pi^0(2)} \sim 2.3 \times 10^{-11}$.

Impact of form factor uncertainties

Very rough description of measurement errors in the double-virtual form factor

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$$
$$\rightarrow \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)\,(1 + \delta_2(Q_1, Q_2))$$

where the momentum dependent errors $\delta_2(Q_1, Q_2)$ are assumed as follows:



Note the unequal bin sizes ! In brackets: number of MC events N_i in each bin $\sim \sigma \sim \mathcal{F}^2_{\pi^0\gamma^*\gamma^*} \Rightarrow \delta\mathcal{F}_{\pi^0\gamma^*\gamma^*} = \sqrt{N_i}/(2N_i)$ (total: 556 events). For lowest bin, assumed error ("extrapolation" from boundary values), no events in simulation (detector acceptance).

Monte Carlo simulations (preliminary) for BES III (Mainz group) based on LMD+V model in EKHARA (Czyż, Ivashyn '11)

Number of events and corresponding precision for $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ should be achievable with current data set plus a few more years of data taking.

For the single-virtual form factor we proceed in a similar way

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0)$$
$$\rightarrow \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0)\,(1 + \delta_1(Q))$$

where we assumed the following momentum dependent errors for $\delta_1(Q)$:

Momentum range [GeV]	$\delta_1(Q)$
$0 \leq Q < 0.5$	5%
$0.5 \leq Q < 2$	7.5%
$2 \leq Q$	12.5%

Based on measurements by CLEO and ongoing analysis by BES III. For lowest bin, assumed error. Use of a dispersion relation for form factor could help at low energies (Hoferichter et al. '14).

Impact of form factor uncertainties (continued)

LMD+V model (for illustration)

For LMD+V FF, region $Q_{1,2} < 0.5$ GeV gives about 59% to total result. For comparison, VMD FF gives 64%.

$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$	Change in %	Comment
$62.9^{+12.1}_{-11.2}$	$^{+19.3}_{-17.8}$	Given δ_1, δ_2
$62.9^{+2.7}_{-2.6}$	$^{+4.3}_{-4.2}$	Lowest bin $Q < 0.5$ GeV in $\delta_1 = 5\%$, rest: $\delta_{1,2} = 0$
$62.9^{+0.7}_{-0.7}$	$^{+1.2}_{-1.0}$	Bins $Q \geq 0.5$ GeV in δ_1 as given, rest: $\delta_{1,2} = 0$
$62.9^{+5.2}_{-5.2}$	$^{+8.3}_{-8.2}$	Lowest bin $Q_{1,2} < 0.5$ GeV in $\delta_2 = 15\%$, rest: $\delta_{1,2} = 0$
$62.9^{+3.1}_{-3.0}$	$^{+4.9}_{-4.8}$	Bins $Q_{1,2} \geq 0.5$ GeV in δ_2 as given, rest: $\delta_{1,2} = 0$
$62.9^{+10.9}_{-10.2}$	$^{+17.3}_{-16.3}$	Given δ_1, δ_2, but lowest bin in δ_1: 5% $\to$ 3%
$62.9^{+9.4}_{-8.7}$	$^{+14.9}_{-13.8}$	Given δ_1, δ_2, but lowest bin in δ_2: 15% $\to$ 7.5%
$62.9^{+9.1}_{-8.5}$	$^{+14.5}_{-13.5}$	In addition: bins in δ_2 with 11% $\to$ 7%

In order to reach goal of 10% error for $a_\mu^{\mathrm{HLbL};\pi^0}$, it would help, if one could measure double-virtual TFF in region $Q_{1,2} < 0.5$ GeV or constrain it using DR.

Note: most model evaluations of $a_\mu^{\mathrm{HLbL};\pi^0}$ (pion-pole, pion-pole with constant FF at external vertex (Melnikov, Vainshtein '04) or pion-exchange with off-shell form factors) agree at the level of 15%, but the full range of estimates is much larger: $a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11} = (65 \pm 15) \times 10^{-11} (\pm 23\%)$.

HLbL @ NLO

Colangelo et al. '14

Recently, a surprisingly large NNLO HVP contribution was obtained by Kurz et al. '14:

$$a_\mu^{\text{HVP, LO}} = (6907.5 \pm 47.2) \times 10^{-11}$$
$$a_\mu^{\text{HVP, NLO}} = (-100.3 \pm 2.2) \times 10^{-11}$$
$$a_\mu^{\text{HVP, NNLO}} = (12.4 \pm 0.1) \times 10^{-11}$$

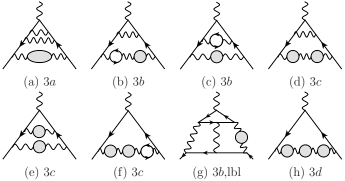
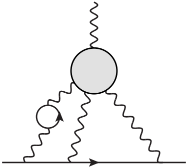


Figure 1: Sample NNLO Feynman diagrams contributing to a_μ^{had}.

Enhancement because of large $\ln(m_\mu/m_e)$, prefactors π^2 in QED LbL.

Could there be a similarly large effect in HLbL @ NLO ?
We calculated the potentially large contribution from an additional electron loop (using simple VMD model for pion-pole to model full HLbL)

$$a_\mu^{\pi^0\text{-pole, NLO}} = 1.5 \cdot 10^{-11}$$



2.6% effect of $a_\mu^{\pi^0\text{-pole}} = 57.2 \cdot 10^{-11}$, very close to renormalization group arguments $3 \times \frac{\alpha}{\pi} \times \frac{2}{3} \log \frac{m_\mu}{m_e} \approx 2.5\%$.

Estimating the not yet calculated diagrams with HLbL with additional radiative corrections to the muon line or internal HLbL by comparing with HLbL insertions with muon loop in QED, which are suppressed by factor 4, we obtain the estimate:

$$a_\mu^{\text{HLbL, NLO}} = (3 \pm 2) \cdot 10^{-11}$$

Negligible with current precision goal for HLbL !

Conclusions

- a_μ: Test of Standard Model, potential window to New Physics:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (278 \pm 88) \times 10^{-11} \qquad [3.1\ \sigma]$$

  Difference can be explained in general MSSM with light sleptons, sneutrinos, charginos, neutralinos ($M_{\rm SUSY} \sim 100 - 300$ GeV) and large $\tan\beta \sim 10 - 30$.

- Two new planned $g-2$ experiments at Fermilab and JPARC with goal of $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$ (factor 4 improvement).
- Theory needs to match this precision !
- Hadronic vacuum polarization

  Ongoing and planned experiments on $\sigma(e^+e^- \to \text{hadrons})$ with a goal of $\delta a_\mu^{\text{HVP}} = (20 - 25) \times 10^{-11}$ (factor 2 improvement).

- Hadronic light-by-light scattering
  - Need a much better understanding of the complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$.
  - Better theoretical models needed; more constraints from theory (ChPT, pQCD, OPE); close collaboration of theory and experiment to measure interactions of hadrons with photons (decays, form factors, cross-sections).
  - Promising new data driven approach using dispersion relations for π^0, η, η' and $\pi\pi$. Still needed: data for scattering of off-shell photons !
    π^0-pole: preliminary analysis using simulations for BES III: could achieve about 20% precision, with a few more years of data taking. Problem: low momentum region $Q_i < 0.5$ GeV: informations from other experiments (KLOE 2 ?, Belle 2 ?) or DR for double-virtual transition form factor needed to reach goal of 10% error.
  - Future: Lattice QCD.

Backup slides

Determination of fine-structure constant α from $g-2$ of electron

- Experimental value (Hanneke et al. '08):

$$a_e^{\mathrm{exp}} = (1\ 159\ 652\ 180.73 \pm 0.28) \times 10^{-12} \quad [0.24\mathrm{ppb}]$$

- Recent measurement of α via recoil-velocity of Rubidium atoms in atom interferometer (Bouchendira et al. '11):

$$\alpha^{-1}(\mathrm{Rb}) = 137.035\ 999\ 037(91) \quad [0.66\mathrm{ppb}]$$

  This leads to (Aoyama et al. '12):

$$a_e^{\mathrm{SM}}(\mathrm{Rb}) = 1\ 159\ 652\ 181.82\ \underbrace{(6)}_{c_4}\ \underbrace{(4)}_{c_5}\ \underbrace{(2)}_{\mathrm{had}}\ \underbrace{(78)}_{\alpha(\mathrm{Rb})}\ [78] \times 10^{-12} \quad [0.67\mathrm{ppb}]$$

$$\Rightarrow a_e^{\mathrm{exp}} - a_e^{\mathrm{SM}}(\mathrm{Rb}) = -1.09(0.83) \times 10^{-12} \quad [\text{Error from } \alpha(\mathrm{Rb}) \text{ dominates !}]$$

  $\rightarrow$ Test of QED !

- Use a_e^{exp} to determine α from series expansion in QED (contributions from weak and strong interactions under control !). Assume: Standard Model "correct", no New Physics (Aoyama et al. '12):

$$\alpha^{-1}(a_e) = 137.035\ 999\ 1657\ \underbrace{(68)}_{c_4}\ \underbrace{(46)}_{c_5}\ \underbrace{(24)}_{\mathrm{had+EW}}\ \underbrace{(331)}_{a_e^{\mathrm{exp}}}\ [342] \quad [0.25\mathrm{ppb}]$$

  The uncertainty from theory has been improved by a factor 4.5 by Aoyama et al. '12, the experimental uncertainty in a_e^{exp} is now the limiting factor.

- Today the most precise determination of the fine-structure constant α, a fundamental parameter of the Standard Model.

HLbL scattering: Summary of selected results

Some results for the various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15±10	22±5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalar-exchanges dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- PdRV: Analyzed results obtained by different groups with various models and suggested new estimates for some contributions (shifted central values, enlarged errors). Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '04 on pseudoscalar-pole contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on off-shell form factors. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Expressions for weight functions $\tilde{I}_{1,2}(Q_1, Q_2, t)$

Jegerlehner, AN '09

$$\tilde{I}_1(Q_1, Q_2, t) = \left(-\frac{2\pi}{3}\right)\sqrt{1-t^2}\,\frac{Q_1^3 Q_2^3}{Q_2^2+m_\pi^2}\,I_1(Q_1, Q_2, t)$$

$$\tilde{I}_2(Q_1, Q_2, t) = \left(-\frac{2\pi}{3}\right)\sqrt{1-t^2}\,\frac{Q_1^3 Q_2^3}{Q_3^2+m_\pi^2}\,I_2(Q_1, Q_2, t)$$

$$\begin{aligned}
I_1(Q_1, Q_2, t) = X(Q_1, Q_2, t)\Big(& 8\,P_1\,P_2\,(Q_1\cdot Q_2) - 2\,P_1\,P_3\,(Q_2^4/m_\mu^2 - 2\,Q_2^2) - 2\,P_1\,(2 - Q_2^2/m_\mu^2 + 2\,(Q_1\cdot Q_2)\,/m_\mu^2) \\
& +4\,P_2\,P_3\,Q_1^2 - 4\,P_2 - 2\,P_3\,(4 + Q_1^2/m_\mu^2 - 2\,Q_2^2/m_\mu^2) + 2/m_\mu^2\Big) \\
& -2\,P_1\,P_2\,(1 + (1-R_{m1})\,(Q_1\cdot Q_2)\,/m_\mu^2) + P_1\,P_3\,(2 - (1-R_{m1})\,Q_2^2/m_\mu^2) + P_1\,(1-R_{m1})/m_\mu^2 \\
& +P_2\,P_3\,(2 + (1-R_{m1})^2\,(Q_1\cdot Q_2)\,/m_\mu^2) + 3\,P_3\,(1-R_{m1})/m_\mu^2
\end{aligned}$$

$$\begin{aligned}
I_2(Q_1, Q_2, t) = X(Q_1, Q_2, t)\Big(& 4\,P_1\,P_2\,(Q_1\cdot Q_2) + 2\,P_1\,P_3\,Q_2^2 - 2\,P_1 + 2\,P_2\,P_3\,Q_1^2 - 2\,P_2 - 4P_3 - 4/m_\mu^2\Big) \\
& -2\,P_1\,P_2 - 3\,P_1\,(1-R_{m2})/(2m_\mu^2) - 3\,P_2\,(1-R_{m1})/(2m_\mu^2) - P_3\,(2 - R_{m1} - R_{m2})/(2m_\mu^2) \\
& +P_1\,P_3\,(2 + 3\,(1-R_{m2})\,Q_2^2/(2m_\mu^2) + (1-R_{m2})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2)) \\
& +P_2\,P_3\,(2 + 3\,(1-R_{m1})\,Q_1^2/(2m_\mu^2) + (1-R_{m1})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2))
\end{aligned}$$

where $Q_3^2 = (Q_1+Q_2)^2$, $\quad Q_1\cdot Q_2 = Q_1 Q_2 \cos\theta$, $\quad t = \cos\theta$

$P_1^2 = 1/Q_1^2$, $\quad P_2^2 = 1/Q_2^2$, $\quad P_3^2 = 1/Q_3^2$, $\quad X(Q_1, Q_2, t) = \frac{1}{Q_1 Q_2\, x}\arctan\left(\frac{zx}{1-zt}\right)$,

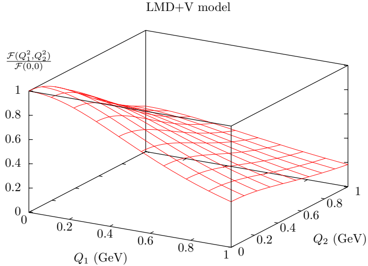
$x = \sqrt{1-t^2}$, $\quad z = \frac{Q_1 Q_2}{4m_\mu^2}(1-R_{m1})(1-R_{m2})$, $\quad R_{mi} = \sqrt{1+4m_\mu^2/Q_i^2}$

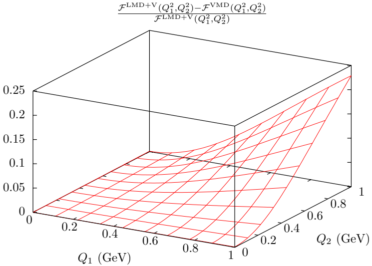
Form factor model: LMD+V (large-N_c QCD) versus VMD

For single-virtual FF, both models give equally good fit to CLEO data. Main difference: double-virtual case. VMD FF violates OPE, falls off too fast. For large Q^2:

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(Q^2, Q^2) \sim \mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(Q^2, Q^2) \sim 1/Q^2$$

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(Q^2, Q^2) \sim 1/Q^4$$





Define: $\Delta\mathcal{F}(Q_1^2, Q_2^2) = \mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2) - \mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

Q_1 [GeV]	Q_2 [GeV]	$\frac{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)}$	$\frac{\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)}$	$\frac{\Delta\mathcal{F}(Q_1^2,Q_2^2)}{\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(Q_1^2,Q_2^2)}$
0.5	0	0.707	0.706	0.0003
1	0	0.376	0.376	0.001
0.5	0.5	0.513	0.499	0.027
1	1	0.183	0.141	0.23

Since LMD+V and VMD FF differ for $Q_1 = Q_2 = 1$ GeV by 23%, it might be possible to distinguish the two models experimentally at BES III, if binning is chosen properly.

The LMD+V form factor

Knecht, AN, EPJC '01; AN '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ in large-N_c QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V).
- $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ fulfills all leading and some subleading QCD short-distance constraints from operator product expansion (OPE).
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance).
- Normalized to decay width $\Gamma_{\pi^0\to\gamma\gamma}$

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\rm LMD+V}(q_1^2, q_2^2) \; = \; \frac{F_\pi}{3} \, \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2) + h_1\, (q_1^2 + q_2^2)^2 + \bar{h}_2\, q_1^2\, q_2^2 + \bar{h}_5\, (q_1^2 + q_2^2) + \bar{h}_7}{(q_1^2 - M_{V_1}^2)\, (q_1^2 - M_{V_2}^2)\, (q_2^2 - M_{V_1}^2)\, (q_2^2 - M_{V_2}^2)}$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

Free model parameters: $h_i, \bar{h}_i$

Transition form factor:

$$F^{\rm LMD+V}(Q^2) = \frac{F_\pi}{3} \, \frac{1}{M_{V_1}^2 M_{V_2}^2} \, \frac{h_1\, Q^4 - \bar{h}_5\, Q^2 + \bar{h}_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}$$

- $h_1 = 0 \text{ GeV}^2$ (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(-Q^2, 0) \sim 1/Q^2$)
- $\bar{h}_2 = -10.63 \text{ GeV}^2$ (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $\bar{h}_5 = 6.93 \pm 0.26 \text{ GeV}^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(-Q^2, 0)$)
- $\bar{h}_7 = -\frac{N_c M_{V_1}^4 M_{V_2}^4}{4\pi^2 F_\pi^2} = -14.83 \text{ GeV}^6$ (or normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

The VMD form factor

Vector Meson Dominance:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^{0*}\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_\pi} \frac{M_V^2}{q_1^2 - M_V^2} \frac{M_V^2}{q_2^2 - M_V^2}$$

Only two model parameters: F_π and M_V

Note:

- VMD form factor factorizes $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = f(q_1^2) \times f(q_2^2)$. This might be a too simplifying assumption / representation.
- VMD form factor has wrong short-distance behavior: $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^4$, for large q^2, falls off too fast compared to OPE prediction $\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^2$.

Transition form factor:

$$F^{\mathrm{VMD}}(Q^2) = -\frac{N_c}{12\pi^2 F_\pi} \frac{M_V^2}{Q^2 + M_V^2}$$

New Physics contributions to the muon $g-2$

Define:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (290 \pm 90) \times 10^{-11} \qquad \text{(Jegerlehner, AN '09)}$$

Absolute size of discrepancy is actually unexpectedly large, compared to weak contribution (although there is some cancellation there):

$$\begin{aligned} a_\mu^{\text{weak}} &= a_\mu^{\text{weak, (1)}}(W) + a_\mu^{\text{weak, (1)}}(Z) + a_\mu^{\text{weak, (2)}} \\ &= (389 - 194 - 41) \times 10^{-11} \\ &= 154 \times 10^{-11} \end{aligned}$$

Assume that New Physics contribution with $M_{\text{NP}} \gg m_\mu$ decouples:

$$a_\mu^{\text{NP}} = \mathcal{C} \frac{m_\mu^2}{M_{\text{NP}}^2}$$

where naturally $\mathcal{C} = \frac{\alpha}{\pi}$, like from a one-loop QED diagram, but with new particles. Typical New Physics scales required to satisfy $a_\mu^{\text{NP}} = \Delta a_\mu$:

$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$\left(\frac{\alpha}{\pi}\right)^2$
M_{NP}	$2.0^{+0.4}_{-0.3}$ TeV	100^{+21}_{-13} GeV	5^{+1}_{-1} GeV

Therefore, for New Physics model with particles in $250 - 300$ GeV mass range and electroweak-size couplings $\mathcal{O}(\alpha)$, we need some additional enhancement factor, like large $\tan\beta$ in the MSSM, to explain the discrepancy Δa_μ.

a_e, a_μ: Dark photon

In some dark matter scenarios, there is a relatively light, but massive "dark photon" A'_μ that couples to the SM through mixing with the photon:

$$\mathcal{L}_{\rm mix} = \frac{\varepsilon}{2} F^{\mu\nu} F'_{\mu\nu}$$

$\Rightarrow$ A'_μ couples to ordinary charged particles with strength $\varepsilon \cdot e$.
$\Rightarrow$ additional contribution of dark photon with mass m_V to the $g-2$ of a lepton (electron, muon) (Pospelov '09):

$$\begin{aligned} a_\ell^{\text{dark photon}} &= \frac{\alpha}{2\pi}\varepsilon^2 \int_0^1 dx \frac{2x(1-x)^2}{\left[(1-x)^2 + \frac{m_V^2}{m_\ell^2} x\right]} \\ &= \frac{\alpha}{2\pi}\varepsilon^2 \times \begin{cases} 1 & \text{for } m_\ell \gg m_V \\ \frac{2m_\ell^2}{3m_V^2} & \text{for } m_\ell \ll m_V \end{cases} \end{aligned}$$

For values $\varepsilon \sim (1-2) \times 10^{-3}$ and $m_V \sim (10-100)$ MeV, the dark photon could explain the discrepancy $\Delta a_\mu = 290 \times 10^{-11}$.

Various searches for the dark photon have been performed, are under way or are planned at BABAR, Jefferson Lab, KLOE, MAMI and other experiments. Essentially all of the parameter space in the (m_V, ε)-plane to explain the muon $g-2$ discrepancy has now been ruled out.
For a recent overview, see: *Dark Sectors and New, Light, Weakly-Coupled Particles* (Snowmass 2013), Essig et al., arXiv:1311.0029 [hep-ph].