Coulomb Collisions in a “Single” Ionized Plasma

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Abstract. The concept of informativeness of plasma physics scenarios is explained. Natural ideas of developing highly informative models of plasma kinetics are spelled out. The current stage of their adaptation to modeling plasma Coulomb collisions is reported. The remaining problems in the creation of a highly informative kinetic model of the collisional evolution of homogeneous unmagnetized plasma are clarified.

INTRODUCTION

The most important aspect of a physical theory is the extent to which its predictions of the behavior of evolving physical systems agree with real pictures of their macrophysical evolutions. More specifically, the longer a theoretical scenario objectively portrays the macrophysical evolution of a system, the better. For a clearer characterization of this aspect of physical-theoretical scenarios, we use the term *informativeness*.

As applied to plasma physical studies, this means that the longer the theoretical scenario adequately depicts the real picture of the plasma macrophysical evolution, the higher the estimate the researcher should suggest for the scenario informativeness. Thus, increasing the informativeness of plasma scenarios should be a major motivation for the development of plasma theory.

However, the fundamentals of traditional theory prevent plasma researchers from success in pursuing this motivation. Basically, the conventional machineries of the theory provide scenarios of inappropriately low informativeness: Most plasma scenarios have an *arbitrary* correspondence with objective pictures of plasma evolutions in respective physical situations. In other words, usual approaches of the theory allow different and even incompatible versions of a specific plasma phenomenon to be generated in an equally rigorous manner. A rich set of illustrations on this point is provided by the nonlinear effects of a weakly turbulent plasma [1–3]. We have long before clarified the reasons of the theory non-informativeness. There are *the fallacious tradition of substituting real plasmas by probabilistic ensembles of plasmas* and *a lack of a proper understanding of the significance of the asymptotic nature of convergence of successive approximations to a plasma scenario*. On the one hand, the interplay of the ensemble statistics (and hence respective deductions on the “*physics*” of the plasma evolution) strongly depends on the ensemble content. On the other hand, the conditional limit of successive approximations to a plasma scenario depends substantially on the choice of the leading order approximation of the iteration procedure, whereat differing limits stand for diverging scenarios. (We emphasize that successive iterations to a plasma scenario are always generated to improve the precision of the scenario. Due to the asymptotic nature of the theory convergence, the improvement of the scenario is inevitably superseded by a reduction in the scenario accuracy after some order of consideration. On this path, the most “precise” scenario depends on the choice of the leading order approximation of the corresponding perturbation theory).

We stress that the above two reasons of theory non-informativeness are inseparable from each other. (Explanations on this point are given in Refs. [4–7]). They dictate the following ideas of developing the most informative of possible plasma scenarios. First, the researcher should refrain from the plasma ensemble substitution. This necessitates modifying the basic concepts of the theory, the plasma particle distribution functions. [Mathematically, the distribution function represents some *statistic* of the distribution of discrete charged particles in the phase space of their positions and momentums ($\boldsymbol{r}$–$\boldsymbol{p}$ phase space). Usual approaches involve developing of such a statistic via ensemble averaging of the Klimontovich's distribution $N_\alpha = \sum_n \delta^3(\boldsymbol{r}-\boldsymbol{r}_n(t))\delta^3(\boldsymbol{p}-\boldsymbol{p}_n(t))$. The only possibility of avoiding this averaging is to replace it by *contextually oriented averaging in the phase space of the positions and momenta of plasma particles*.] Second, the researcher should develop successive iterations of the scenario using the direct time integration of intermediate evolution equations. It is this approach that allows one to account properly for available information on the current plasma state and its recent history, and simultaneously to diminish the effect of indeterminate information on temporally remote phases of the plasma evolution.

In this paper we shall analyze the problem of developing a highly informative kinetic model of plasma Coulomb collisions. We shall focus on the traditional situation with homogeneous thermodynamically nonequilibrium plasma out of any leading magnetic field. We first comment former theoretical considerations of this phenomenon. Then we shall clarify the current understanding of problems of developing the highly informative scenario of plasma evolution due to the Coulomb collisions.

HISTORICAL CONCEPTS OF COULOMB COLLISIONS

Landau was the first to consider the Coulomb collisions of charged particles in a plasma [8]. He has adapted the Boltzmann's gas kinetic theory [9] to the case of a fully ionized gas. This yielded an equation for modeling the collisions of charged particles of the gas. It is known as a plasma kinetic equation with the Landau collision integral; the derivation of this equation has initiated the development of plasma kinetic theory.

The Landau collision integral diverges for both small and large impact parameters. Accordingly, it was agreed to artificially cut off the corresponding integration at its top and bottom limits. A more logical model of Coulomb collisions was developed by Lenard [10] and independently by Balescu [11]. Their equation is characterized as *an equation accounting for the dynamic plasma polarization*. Its collision integral does not diverge at large impact parameters. The lines of reasoning of Lenard and Balescu were notably differing. Still, the basis of both the Lenard's and the Balescu's plasma kinetic considerations, i.e., the research orientation toward evolving continuous probabilistic ensembles of plasmas, has ensured the coincidence of their results. We would like to add that the derivations of the Boltzmann gas kinetic equation and the Landau collision integral were also oriented conceptually to ensembles of physical systems rather than to original systems (gas and ionized plasma, respectively). Thus, the concept of plasma ensemble underlies all historically known models of Coulomb collisions. Correspondingly, the informativeness of respective plasma scenarios is doubtful.

LOGICS OF MOTION TO A HIGHLY INFORMATIVE KINETIC MODEL OF COULOMB COLLISIONS AND ITS FIRST MILESTONES

In our kinetics, the statistic $f_\alpha(\boldsymbol{p}, t)$ of distribution of given particle species α evolves according to the equation

$$\frac{\partial}{\partial t} f_\alpha = \frac{e_\alpha}{c} v_i \frac{\partial}{\partial p^\beta} \left\langle \delta N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t) F^{i\beta}(\boldsymbol{r}, t) \right\rangle . \tag{1}$$

Here $\mathbf{F}(\boldsymbol{r}, t)$ is the electromagnetic field tensor (EMF-tensor) that is exclusively microstructured in our problem. Angle brackets denote the averaging over large parallelepiped-shaped volumes of phase space with small dimensions in momentum components (i.e., over usual space in fact). The function $f_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)$ is the mean of the Klimontovich's distribution over respective volume. The right-hand side of the equation contains a special case of the *two-point correlation function* $\left\langle \delta N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t) F^{ij}(\boldsymbol{r}', t') \right\rangle$. The difference of the spatial variables under the averaging symbol in the latter is supposed to be fixed: the couple of respective variables changes synchronously in the averaging. Accordingly, the notation of spatial dependencies is somewhat redundant (see an expanded footnote on this point in paper 12). Tensor subscripts and superscripts are used for ease of interpreting formulae.

The two-point correlation function is advanced in time by the *two-time correlation function* which is the mean product of two electromagnetic field tensors $\boldsymbol{\Phi}(\boldsymbol{r}, t, \boldsymbol{r}', t') = \langle \mathbf{F}(\boldsymbol{r}, t) \otimes \mathbf{F}(\boldsymbol{r}', t') \rangle$. We shall write a linear approximation to the respective evolution equation. Similar to predecessors, we assume that the potential part only of microstructural electric fields in a plasma is important. Then one obtains

$$\left[\frac{\partial}{\partial t} + v^\varepsilon \frac{\partial}{\partial r^\varepsilon} \right] \left\langle \delta N_\alpha(\boldsymbol{r}, \boldsymbol{p}, t) F^{0\beta}(\boldsymbol{r}', t') \right\rangle = e_\alpha \frac{\partial f_\alpha(t)}{\partial p^\gamma} \Phi^{0\gamma 0\beta}(\boldsymbol{r}, t, \boldsymbol{r}', t'). \tag{2}$$

Let us introduce spatial Fourier transforms:

$$\langle\delta N_\alpha \mathbf{F}\rangle_{\boldsymbol{k}}(\boldsymbol{p},t,t') = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{R}\,\langle\delta N_\alpha(\boldsymbol{r}+\boldsymbol{R}/2,\boldsymbol{p},t)\mathbf{F}(\boldsymbol{r}-\boldsymbol{R}/2,t')\rangle\exp\left(-i\left(\boldsymbol{k}\cdot\boldsymbol{R}\right)\right), \tag{3}$$

$$\boldsymbol{\Phi}_{\boldsymbol{k}}(t,t') = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{R}\boldsymbol{\Phi}(\boldsymbol{r}+\boldsymbol{R}/2,t,\boldsymbol{r}-\boldsymbol{R}/2,t')\exp\left(-i\left(\boldsymbol{k}\cdot\boldsymbol{R}\right)\right). \tag{4}$$

Note that the resulting tensors have simple structures: $\left\langle\delta N_\alpha F^{0\beta}\right\rangle_{\boldsymbol{k}} = \left(k^\beta/k\right)[\delta N_\alpha F]_{\boldsymbol{k}}$ and $\Phi^{0\gamma0\beta}_{\boldsymbol{k}} = \left(k^\beta k^\gamma/k^2\right)\Phi_{\boldsymbol{k}}$. Equation (2) yields the evolution equation of scalar $[\delta N_\alpha F]_{\boldsymbol{k}}$,

$$\left[\frac{\partial}{\partial t} + i\left(\boldsymbol{k}\cdot\boldsymbol{v}\right)\right][\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t,t') = \frac{e_\alpha}{k}\left(\boldsymbol{k}\cdot\frac{\partial f_\alpha(t)}{\partial\boldsymbol{p}}\right)\Phi_{\boldsymbol{k}}(t,t'). \tag{5}$$

An independent relation between the two-point and two-time correlation functions constitutes an analog of the usual equation for the divergence of the potential electric field:

$$\Phi_{\boldsymbol{k}}(t,t') = -\frac{4\pi i}{k}\sum_\alpha e_\alpha\int[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t,t')\,d^3\boldsymbol{p}. \tag{6}$$

Formally, knowledge of the initial data $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t')$ would have permitted one to construct a solution to simultaneous equations (5,6) for all $t \geqslant t'$. However, the initial data $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t')$ are never known to full extent and cannot be defined on the basis of linear equations (5,6). Some easy understanding can be obtained from considerations independent of these equations only for the "short-wavelength" limit of $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t')$ and also for the respective behavior of $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t,t')$ during a rather small time delay $t-t'$,

$$\left.\begin{aligned}[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t') &= -ie_\alpha/\left(2k\pi^2\right)f_\alpha(\boldsymbol{p},t'),\\ \Phi_{\boldsymbol{k}}(t,t) &= 4n_0e^2/\left(\pi k^2\right)\end{aligned}\right\},\quad k \gtrsim (n_0)^{1/3} \gg 1/r_D, \tag{7}$$

$$\left.\begin{aligned}[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t,t') &= -ie_\alpha/\left(2k\pi^2\right)f_\alpha(\boldsymbol{p},t')\exp\left(-i\left(\boldsymbol{k}\cdot\boldsymbol{v}\right)(t-t')\right),\\ \Phi_{\boldsymbol{k}}(t,t') &= 2e^2/\left(\pi k^2\right)\sum_\alpha\int d^3\boldsymbol{p}f_\alpha(\boldsymbol{p},t')\exp\left(-i\left(\boldsymbol{k}\cdot\boldsymbol{v}\right)(t-t')\right)\end{aligned}\right\},\quad k \gtrsim (n_0)^{1/3}, 0 < t-t' \ll 1/\left(kv_{Te}\right). \tag{8}$$

(Here n_0 is the plasma density). Meanwhile, just the full set of data $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t')$ is necessary to calculate the right-hand side of Equation (1). [Indeed, we have $\left\langle\delta N_\alpha(\boldsymbol{r},\boldsymbol{p},t)F^{0\beta}(\boldsymbol{r},t)\right\rangle = \int d^3\boldsymbol{k}\left(k^\beta/k\right)[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t,t)$.] That is, one should calculate $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t',t')$ for all values of k, from $k=0$ to $k \lesssim (n_0)^{1/3}$. [The larger values of k do not contribute to the right-hand side of equation (1), in view of the isotropic structure of $[\delta N_\alpha F]_{\boldsymbol{k}}(t',t')$ at respective wave vectors.] Undoubtedly, the required data can be deduced from the known short-wavelength limit (7) only through consideration of the proper *nonlinear* problem. (We comment that the nonlinear interaction of the short-wavelength harmonics of $[\delta N_\alpha F]_{\boldsymbol{k}}$ generates the motive force that advances in time the harmonics with intermediate k).

In Figure 1 we present the first corrected expression of the two-point correlation function in terms of the two-time correlation functions. It is obtained as follows. We have taken account of the lowest order correction to the right-hand side of linear equation (5). The corrected equation was integrated over the entry time t from some moment t_0 that is rather remote in the past from both t and the exit time t'. We then set $t_0 = -\infty$ and omit the term with $[\delta N_\alpha F]_{\boldsymbol{k}}(\boldsymbol{p},t_0,t')$. The point is that when the time delay $\min(t,t')-t_0$ is great compared to the period of Langmuir oscillations, the effect of this term is negligible due to the intrinsic "phase mixing" of the term within the right-hand side of Equation (1).

The following graphical notation is used (see Refs. [1–3, 13] for more details). The thin solid line denotes the bare Green function of a given species of plasma particles ${}^0G_\alpha(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}',\boldsymbol{p}',t')$, which is the solution to the equation

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + v^\beta\frac{\partial}{\partial r^\beta} + \frac{e_\alpha}{c}v_i{}^0F^{i\beta}\frac{\partial}{\partial p^\beta}\right]{}^0G_\alpha(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}',\boldsymbol{p}',t')\\ &\qquad = \delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t').\end{aligned} \tag{9}$$

The dashed line denotes the operator of the electromagnetic Green function $\mathcal{F}$ which corresponds conceptually to the well-known delayed potentials. This function is a definite integral differential operator that yields the expression of the

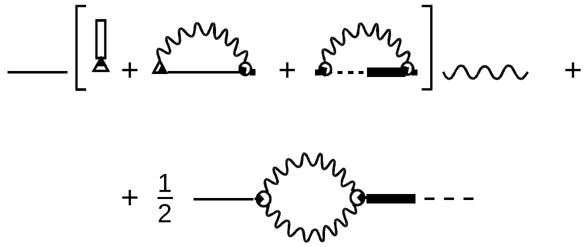

FIGURE 1. Expression of the *two-point* correlation function $\langle \delta N \delta \mathbf{F} \rangle$ in terms of the *two-time correlation* functions $\mathbf{\Phi}$ (wavy lines).

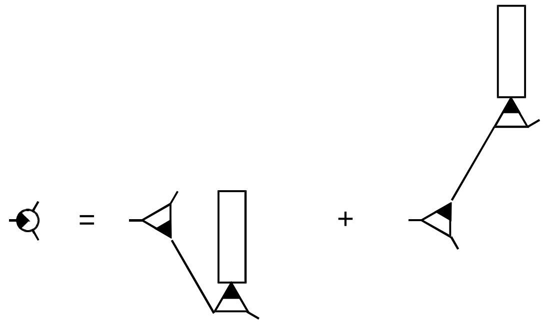

FIGURE 2. Graphical definition of the symmetric vertex ${}^{\alpha}\mathcal{H}^{ijkl}$. (The pairs of superscripts i, j and k, l correspond to the two exit ends by which the vertex is connected to the $\mathcal{F}$-ends of the other graphical constructions.)

EMF tensor in terms of the charge and the charge current densities. That is, in the absence of external electromagnetic radiation, the EMF tensor in the plasma is

$$F_{ik}(\boldsymbol{r}, t) = {}^{0}F_{ik}(\boldsymbol{r})$$

$$+ \sum_{\alpha} e_{\alpha} \int_{-\infty}^{t} dt' \int \mathcal{F}_{ik}(\boldsymbol{r}, t, \boldsymbol{r}', \boldsymbol{v}', t') N_{\alpha}(\boldsymbol{r}', \boldsymbol{p}', t') d^{3}\boldsymbol{r}' d^{3}\boldsymbol{p}'. \tag{10}$$

Here the tensor ${}^{0}F_{ik}(\boldsymbol{r})$ corresponds to the external stationary magnetic field that is absent in our problem. We stress that the operator $\mathcal{F}$ is the basic mean for accounting the effects of the electromagnetic field generated by plasma particles in the more general case of a plasma subjected to an external electromagnetic radiation.

The explicit form of $\mathcal{F}$ is not needed for our considerations.

Both the bare Green function and the electromagnetic Green function satisfy the causality principle: at $t < t'$, they are identically zero.

One more notation is for the symmetric vertex ${}^{\alpha}\boldsymbol{\mathcal{H}}$ defined by the Equation in Figure 2. In the latter, the oblong rectangle denotes the distribution function f_{α}, and the asymmetric vertex is used following Figure 3. With the asymmetric vertex we associate the time moment t, the momentum $\boldsymbol{p}$, the space position $\boldsymbol{r}$, and the coefficient $-e_{\alpha}$. When the vertex occurs inside the diagram, integration over the respective dummy variables is supposed.

Finally, we use the renormalized Green function $G_{\alpha\alpha'}(\boldsymbol{r}, \boldsymbol{p}, t, \boldsymbol{r}', \boldsymbol{p}', t')$. It is shown graphically by a thick solid line and is defined as the solution to the Dyson type equation in Figure 4.

The first term in the square brackets on the top line of Figure 1 depicts the linear piece of the expression of the two-point function in terms of the two-time functions that complies with Equation (2) and Equation (5).

Expression from Figure 1 can be used for constructing the "current density" that advances in time the two-time correlation function. The corresponding analog of the macroscopic equation $\partial \boldsymbol{E}/\partial t + 4\pi \boldsymbol{j} = 0$ is expressed in terms of our scalars as

$$\frac{\partial}{\partial t} \Phi_{\boldsymbol{k}}(t, t') = -4\pi \int_{-\infty}^{t} dt_{1} \sigma_{\boldsymbol{k}}(t, t_{1}) \Phi_{\boldsymbol{k}}(t_{1}, t') - \mathcal{B}_{\boldsymbol{k}}(t, t'). \tag{11}$$

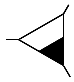

FIGURE 3. Asymmetric vertex. The upper exit is always connected to either the dashed line of $\mathcal{F}$ or the wavy line of the two-time correlation function. The related "EMF-end" $\mathcal{F}^{i\beta}$ of the object is multiplied by v_i and by the momentum derivative $\partial/\partial p^\beta$ of the function connected to the lower exit. The latter exit is marked in black to make analytical interpretation of the diagrams easier.

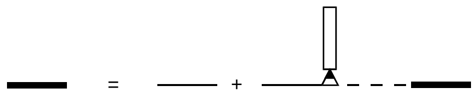

FIGURE 4. Dyson type equation for the renormalized Green function $G_{\alpha\alpha'}(\boldsymbol{r},\boldsymbol{p},t,\boldsymbol{r}',\boldsymbol{p}',t')$.

Here $\sigma_{\boldsymbol{k}}(t,t')$ is a scalar function of *conductivity* (its graphic analog precedes the rightmost wavy line of the two-point correlation function in the top line in Figure 1), and $\mathcal{B}_{\boldsymbol{k}}(t,t')$ depicts the contribution of the second line in Figure 1. Both functions $\Phi_{\boldsymbol{k}}(t,t')$ and $\mathcal{B}_{\boldsymbol{k}}(t,t')$ decay with increasing $t > t'$, due to the temporal loss in correlations of microfields.

Equation (11) should be used to iteratively express the two-time function $\Phi_{\boldsymbol{k}}(t,t')$ in terms of its unknown initial data $\Phi_{\boldsymbol{k}}(t',t')$ that possess the short-wavelength limit (7). After this, the two-time function should be substituted into the analytical analog of the expression of the two-point function $[\delta N_\alpha F]_{\boldsymbol{k}}$ in Figure 1. Subsequently addressing Equation (6), one can consider simultaneous equations (1,11) for unrolling the coordinated evolutions of the spectrum $\Phi_{\boldsymbol{k}}(t',t')$ and the distribution functions $f_\alpha(\boldsymbol{p})$. This will complete the description of the plasma evolution due to the Coulomb collisions.

Note that just equation (11) was used to model the three-wave interactions of potential waves in weakly turbulent plasmas. In respective problems, the characteristic time of decay of $\Phi_{\boldsymbol{k}}(t,t')$ with increasing $t-t'$ is significantly larger than that of the function $\mathcal{B}_{\boldsymbol{k}}(t,t')$. This permits the direct integration of equation (11) at large time delays $t-t'$ and the subsequent development of the time derivative of the basic multiplier in the solution, the wave spectral density $n_{\boldsymbol{k}}$. [This density corresponds factually to an *autocorrelation function* $\Phi_{\boldsymbol{k}}(t,t)$.] The situation with Coulomb collisions is much more complicated. This time, with the bottom bound of the typical wave vector k being $1/r_D$, the functions $\mathcal{B}_{\boldsymbol{k}}$ and $\Phi_{\boldsymbol{k}}$ seem to possess typical times of decay in $t-t'$ of the same order $1/(kv_{Te})$. Therefore, here the procedure developed for plasma turbulence cannot be used to solve the equation, and different ideas for necessary iterations should be advanced. Unfortunately, we have not yet succeeded in formulating a respective leading order solution to the equation and corresponding iteration procedure. This issue requires a more profound study.

REFERENCES

[1] V. Erofeev, *High-informative plasma theory* (LAP, Saarbrücken, 2011).
[2] V. Erofeev, *Printsipy razrabotki visokoinformativnikh modelei plazmennoi kinetiki* (Publishing House of the Siberian Branch of RAS, 2013).
[3] V. I. Erofeev, Physics of Plasmas **22**, p. 092302 (2015).
[4] V. I. Erofeev, J. Plasma and Fusion Research Series **8**, 59–64 (2009).
[5] V. I. Erofeev, Fusion Science and Technology **59**, 316–319 (2011).
[6] V. I. Erofeev, J. Fusion Energy **29**, 337–346 (2010).
[7] V. I. Erofeev, J. Fusion Energy **30**, 157–168 (2011).
[8] L. D. Landau, Phys. Z. Sowjetunion **10**, 154–164 (1936).
[9] L. Boltzmann, Wiener Berichte **66**, 275–370 (1872).
[10] A. Lenard, Ann. Phys. **10**, 390–400 (1960).
[11] R. Balescu, Phys. Fluids **3**, 52–63 (1960).
[12] V. I. Erofeev, Journal of Plasma Physics **80**, 289–3184 (2014).
[13] V. I. Erofeev, Journal of Plasma Physics **57**, 273–298 (1997).