Pseudoscalar-exchange contribution to $(g-2)_\mu$ from rational approximants — and connection to $P \to \bar{\ell}\ell$ decays —

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17th June 2015, PHOTON 2015, Budker Institute of Nuclear Physics

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

PRISMA

Outline

Section 1

Motivation and Introduction to the Method

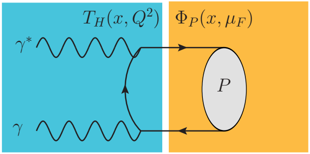
The problem: a first principle QCD description for the TFF

HIGH ENERGIES (pQCD)

- Space-like (SL) $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$

$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \int dx\, T_H(x, Q_i^2, \mu)\Phi_P(x, \mu_F)$$

- $\phi_P(x, \mu_F)$ non-pert. $\to$ **MODELLED!**
- $T_H(x, Q_i^2)$ perturbative in $\alpha_s(Q_i^2)$



$F_{\pi\gamma\gamma^*}(0,\infty)=2F_\pi Q^{-2}$
$F_{\pi\gamma^*\gamma^*}(\infty,\infty)=(2/3)F_\pi Q^{-2}$

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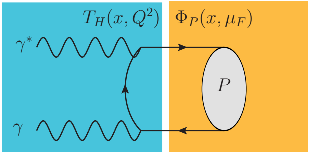

$F_{\pi\gamma\gamma^*}(0,\infty) = 2F_\pi Q^{-2}$
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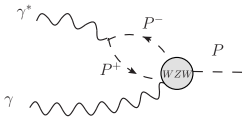
LOW ENERGIES (χPT)

- ABJ anomaly prediction $F_{P\gamma\gamma}(0,0)$
- Extensions for $Q_i^2 > 0$ poor (vectors)
- **MODEL** the vectors (i.e.: $R\chi PT$)



$F_{\pi\gamma\gamma}(0,0) = (4\pi^2 F_\pi)^{-1}$

Objectives and strategies

—What do we want?

A model-independent approach for pseudoscalar transition form factors

Input in HLbL calculations —talk by A. Nyffeler

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Generalize our approach to bivariate functions: Chisholm Approximants

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Padé Approximants: Introduction to the method

Given a function with known series expansion

$$F_{P\gamma\gamma^*}(Q^2) = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ...)$$

Its Padé approximant is defined as

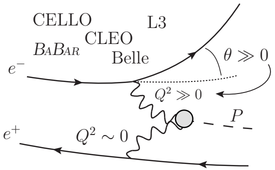
$$P^N_M(Q^2) = \frac{T_N(Q^2)}{R_M(Q^2)} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + c_P Q^4 + ... + \mathcal{O}(Q^2)^{N+M+1})$$

Convergence th. $\Rightarrow$ **Model-independency**
Increase$\{N, M\}$ $\Rightarrow$ **Systematic error estimation**

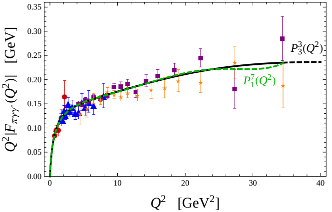
$$P^0_1 = \frac{F_{P\gamma\gamma^*}(0)}{1 - b_P Q^2} = F_{P\gamma\gamma^*}(0)(1 + b_P Q^2 + ... + \mathcal{O}(Q^4)) \quad \cancel{VMD}$$

Correct low energy implementation!

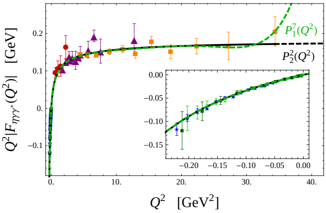
Padé Approximants: Results



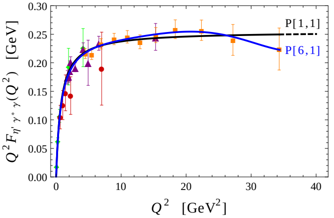
π



η



η'



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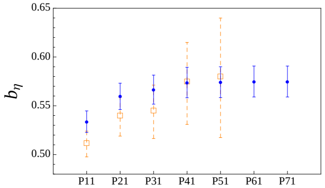

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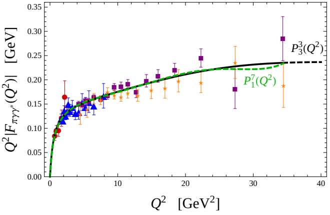

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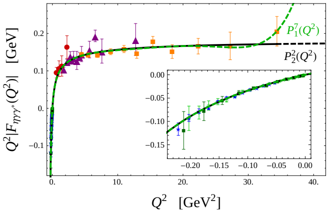
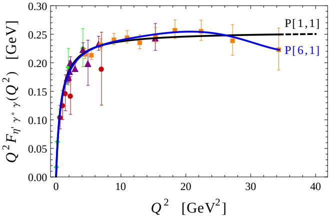


η'



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What about the double virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

Extend Padé approximants to bivariate case (Chisholm '73)

$$C_1^0(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{1,1})(Q_1^2 Q_2^2)}$$

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—Properties

1.Reproduce original series expansion $\Rightarrow$ low energies

$$C_1^0(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)(1 + b_P(Q_1^2 + Q_2^2) + a_{1,1} Q_1^2 Q_2^2 + ...)$$

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2.Reduce to Padé Approximants (already determined)

$$C_1^0(Q^2, 0) = \frac{F_{P\gamma\gamma}(0,0)}{1 - b_P Q^2} = P_1^0(Q^2)$$

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$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{1 + b_P(Q_1^2 + Q_2^2)}; \quad (a_{1,1} \equiv 2b_P^2) \; OPE\checkmark$$

What about the double virtual $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$?

Extend Padé approximants to bivariate case (Chisholm '73)

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$$C_1^0(Q_1^2, Q_2^2)|_{OPE} = \frac{F_{P\gamma\gamma}(0,0)}{(1 + b_P Q_1^2)(1 + b_P Q_2^2)}; \quad (a_{1,1} \equiv b_P^2) \ \textit{Factorization}$$

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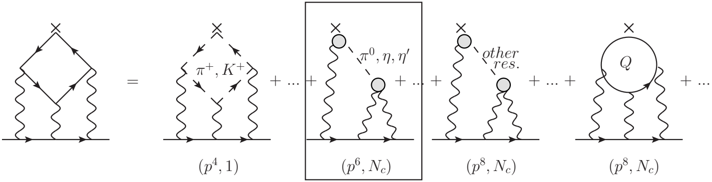
Parameter $a_{1,1}$ from data $\Rightarrow$ Low-energies ... But not available!
Take generous range $a_{1,1} \in \{0 \div 2b_P^2\}$ (includes OPE, fact)

Section 2

Results for $(g-2)_\mu$

$(g-2)_\mu$: hadronic light-by-light



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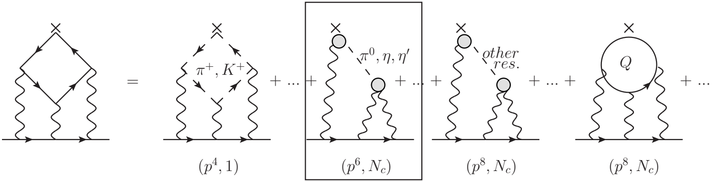
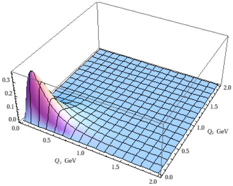




Knecht & Nyffeler: π^0, η, η'-exchange

- Loop integral involving $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$
- SL low energy regime $\to$ our PAs are good
- Multiscale: low-high energies

$(g-2)_\mu$: **hadronic light-by-light**

Our Results from Bivariate Padé Approximants

Units of 10^{-10}	π^0	η	η'	Total
$a_{1,1} = 2b_P^2$ [OPE]	6.64(33)	1.69(6)	1.61(21)	$9.94(40)_{stat}(50)_{sys}$
$a_{1,1} = b_P^2$ [Fact]	5.53(27)	1.30(5)	1.21(12)	$8.04(30)_{stat}(40)_{sys}$
$a_{1,1} = 0$	5.10(23)	1.16(7)	1.07(15)	$7.33(28)_{stat}(37)_{sys}$

$$a_\mu^{HLbL;P} = (9.94(40)(50) \div 7.33(28)(37)) \times 10^{-10} \quad (1.6\sigma^{FLab})$$

Big uncertainty from double-virtual term often non-considererd
High-energies *vs.* Low-energies

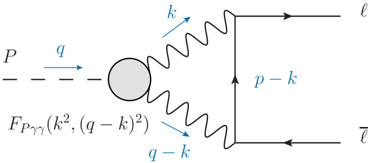
To be compared with pseudoscalar-pole contributions in the literature
BPP: 8.5(1.3); **HKS**: 8.6(0.6); **KN**: 8.3(1.2)

Section 3

Connection to $P \to \bar{\ell}\ell$

$P \to \bar{\ell}\ell$ decays: Introduction

At LO in α_{EM}, this process occurs via 2γ intermediate state.



P. Masjuan, P. Sanchez, arXiv:1504:07001

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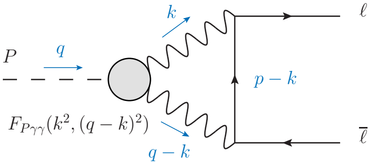

$$\frac{BR(P \to e^+e^-)}{BR(P \to \gamma\gamma)} = 2\left(\frac{\alpha m_\ell}{\pi m_P}\right)^2 \beta_\ell(m_P^2)|\mathcal{A}(m_P^2)|^2,$$

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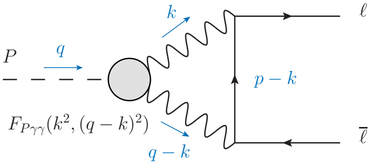

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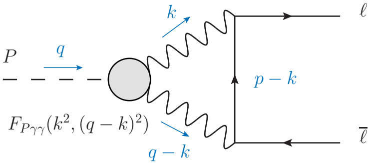

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Again peaked at low (mainly) SL energies!

P. Masjuan, P. Sanchez, arXiv:1504:07001

Case I: $\pi^0 \to e^+e^-$

There has ben a lot of activity since the latest experimental result

$$BR^{KTeV}(\pi^0 \to e^+e^-) = 7.48(38) \times 10^{-8}$$
$$BR^{Th.}(\pi \to e^+e^-) = 6.23(09) \times 10^{-8}$$

Which represents a **3σ deviation**

P. Masjuan, P. Sanchez, arXiv:1504:07001

Case I: $\pi^0 \to e^+e^-$

Taking into account last radiative corrections results —Husek *et al*. '14

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Still, **no model can reproduce such value**
$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)$ enters in HLbL $\Rightarrow$ impact?

What have to say our approximants?

— Use the simplest approximant —

$$\tilde{C}_1^0(Q_1^2, Q_2^2) = \frac{1}{1 + b_P(Q_1^2 + Q_2^2) + (2b_P^2 - a_{1,1})Q_1^2 Q_2^2}$$

P. Masjuan, P. Sanchez, arXiv:1504:07001

Case I: $\pi^0 \to e^+e^-$

Our Result

$$BR(\pi^0 \to e^+e^-) = (6.20 \div 6.41)(5) \times 10^{-8}; a_{1,1} \in \{2b_P^2 \div 0\}$$

Accepted value: $6.23(9) \times 10^{-8}$(Dorokhov et.al. '07) $\Rightarrow$ UNDERESTIMATED

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Hypothetic Double-Virtual Data below 1GeV 30% Error

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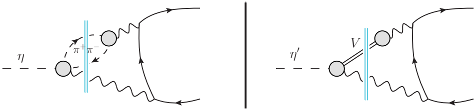
Fix $a_{1,1}$ To Experiment $\Rightarrow (g-2)_\mu$ Impact?

$$a_\mu^{HLbL;\pi^0} = (5.10 \div 6.64)10^{-10} \Rightarrow 2.85 \times 10^{-10} \quad (1.8\sigma^{FLab})$$

P. Masjuan, P. Sanchez, arXiv:1504:07001

Case II: $\eta(\eta') \to \bar{\ell}\ell$ (Work in Progress)

Integral is sensitive to time-like up to m_P^2



P. Masjuan, P. Sanchez, In preparation

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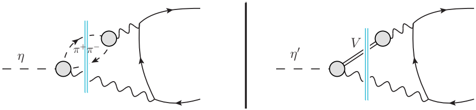
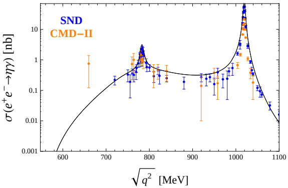
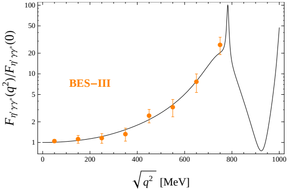
Unitary Bound, $|\mathcal{A}|^2 \geq Im(\mathcal{A})^2_{\gamma\gamma}$, BREAKS



First $Im(\mathcal{A})$ estimation ever: realistic (toy)models





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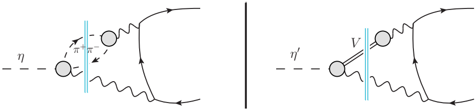

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$$\eta : Im(\pi\pi)/Im(\gamma\gamma) = -0.5\% \quad \checkmark$$
$$\eta' : Im(\rho,\omega)/Im(\gamma\gamma) = -20\% \quad \times$$

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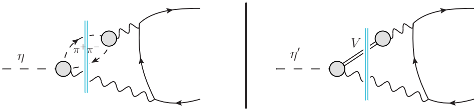

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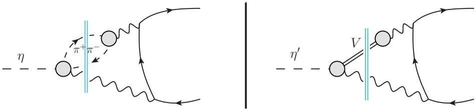
P. Masjuan, P. Sanchez, In preparation

Case II: $\eta(\eta') \to \bar{\ell}\ell$ (Work in Progress)

Unitary Bound, $|\mathcal{A}|^2 \geq Im(\mathcal{A})^2_{\gamma\gamma}$, BREAKS



First $Im(\mathcal{A})$ estimation ever: realistic (toy) models

$$\eta : Im(\pi\pi)/Im(\gamma\gamma) = -0.5\% \quad \checkmark$$
$$\eta' : Im(\rho,\omega)/Im(\gamma\gamma) = -20\% \quad \times$$

For the η negligible: take C_1^0 and include a syst. (1%) error
For the η' combine C_1^0 and resonance description

P. Masjuan, P. Sanchez, In preparation

Case II: $\eta \to \bar{\ell}\ell$

Our C_1^0 Result [exact] - (Preliminary Results)

$$\eta \to e^+e^- = (5.31 \div 5.44)\left(^{+4}_{-5}\right)10^{-9}$$
$$\eta \to \mu^+\mu^- = (4.52 \div 4.72)\left(^{+4}_{-8}\right)10^{-6}$$

P. Masjuan, P. Sanchez, In preparation

Case II: $\eta \to \bar{\ell}\ell$

Our C_1^0 Result [exact] - (Preliminary Results)

$$\eta \to e^+e^- = (5.31 \div 5.44)(^{+4}_{-5})10^{-9}$$
$$\eta \to \mu^+\mu^- = (4.52 \div 4.72)(^{+4}_{-8})10^{-6}$$

Accepted values [approximated]: Dorokhov '10

$$\eta \to e^+e^- = 4.53(9) \times 10^{-9}$$
$$\eta \to \mu^+\mu^- = 5.35(27) \times 10^{-6}$$

P. Masjuan, P. Sanchez, In preparation

Case II: $\eta \to \bar{\ell}\ell$

OUR C_1^0 RESULT [EXACT] - (PRELIMINARY RESULTS)

$$\eta \to e^+e^- = (5.31 \div 5.44)(^{+4}_{-5})10^{-9}$$
$$\eta \to \mu^+\mu^- = (4.52 \div 4.72)(^{+4}_{-8})10^{-6}$$

ACCEPTED VALUES [APPROXIMATED]: Dorokhov '10

$$\eta \to e^+e^- = 4.53(9) \times 10^{-9}$$
$$\eta \to \mu^+\mu^- = 5.35(27) \times 10^{-6}$$

COMPARE TO EXPERIMENT

$$\eta \to e^+e^- \leq 2.3 \times 10^{-6} \text{ HADES '14}$$
$$\eta \to \mu^+\mu^- = 5.8(8) \times 10^{-6} \text{ SATURNE-II '94}$$

P. Masjuan, P. Sanchez, In preparation

Case II: $\eta' \to \bar{\ell}\ell$

Our C_1^0 & Combined Results [exact] - Preliminary Results

$$\eta' \to e^+e^- = (1.82 \div 1.86)(7)10^{-10} \xrightarrow{C_1^0+Res} (1.73 \div 1.77)(7)10^{-10}$$
$$\eta' \to \mu^+\mu^- = (1.36 \div 1.49)(5)10^{-7} \xrightarrow{C_1^0+Res} (1.22 \div 1.35)(5)10^{-7}$$

P. Masjuan, P. Sanchez, In preparation

Case II: $\eta' \to \bar{\ell}\ell$

Our C_1^0 & Combined Results [exact] - Preliminary Results

$$\eta' \to e^+e^- = (1.82 \div 1.86)(7)10^{-10} \xrightarrow{C_1^0+Res} (1.73 \div 1.77)(7)10^{-10}$$
$$\eta' \to \mu^+\mu^- = (1.36 \div 1.49)(5)10^{-7} \xrightarrow{C_1^0+Res} (1.22 \div 1.35)(5)10^{-7}$$

Accepted values [approximated]: Dorokhov '10

$$\eta \to e^+e^- = 1.182(14) \times 10^{-10}$$
$$\eta \to \mu^+\mu^- = 1.364(10) \times 10^{-7}$$

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Case II: $\eta' \to \bar{\ell}\ell$

Our C_1^0 & Combined Results [exact] - Preliminary Results

$$\eta' \to e^+e^- = (1.82 \div 1.86)(7)10^{-10} \xrightarrow{C_1^0+Res} (1.73 \div 1.77)(7)10^{-10}$$
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Accepted values [approximated]: Dorokhov '10

$$\eta \to e^+e^- = 1.182(14) \times 10^{-10}$$
$$\eta \to \mu^+\mu^- = 1.364(10) \times 10^{-7}$$

Compare to Experiment

$$\eta' \to e^+e^- \leq 5.6 \times 10^{-9} \text{ (SND+CMD-III) '15}$$
$$\eta' \to \mu^+\mu^- = \text{—}$$

P. Masjuan, P. Sanchez, In preparation

Conclusions & Outlook

- Rational Approximants have been used to describe the TFF
- Is data driven: better data, better description and easy to apply
- Precise low-energies but QCD constraints as well
- We calculated $(g-2)_\mu^{HLbL;P}$ and $P \to \bar{\ell}\ell$ with systematics
- $\pi \to e^+e^-$, $\eta \to \mu^+\mu^-$ discrepancy $\Rightarrow (g-2)_\mu^{HLbL;P}$, New Phys.?
- $\gamma^*\gamma^* \to P$ (allows $C_1^0 \to C_2^1$) and $P \to \bar{\ell}\ell$ required

- Future: pQCD matching, including cuts and resonance appropriately

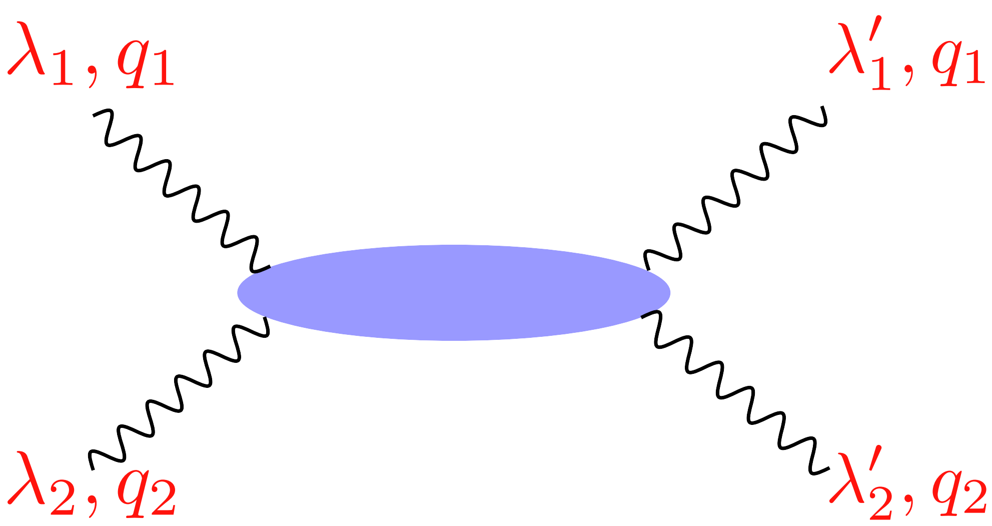
Hadronic light-by-light sum rules

Marc Vanderhaeghen

conference, time

location

sum rules for LbL scattering (I)



$$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \to \gamma^*(\lambda_1', q_1) + \gamma^*(\lambda_2', q_2)$$

kinematical invariants:

$$s = (q_1 + q_2)^2, \qquad u = (q_1 - q_2)^2$$

$$\nu \equiv \frac{s-u}{4}, \quad Q_1^2 \equiv -q_1^2, \quad Q_2^2 \equiv -q_2^2$$

helicity amplitudes: $M_{\lambda_1'\lambda_2', \lambda_1\lambda_2}(\nu, Q_1^2, Q_2^2)$ $\qquad \lambda = 0, \pm 1$

discrete symmetries:

$$P: \quad M_{\lambda_1'\lambda_2', \lambda_1\lambda_2} = M_{-\lambda_1' -\lambda_2', -\lambda_1 -\lambda_2}$$

$$T: \quad M_{\lambda_1'\lambda_2', \lambda_1\lambda_2} = M_{\lambda_1\lambda_2, \lambda_1'\lambda_2'}$$

8 independent amplitudes:

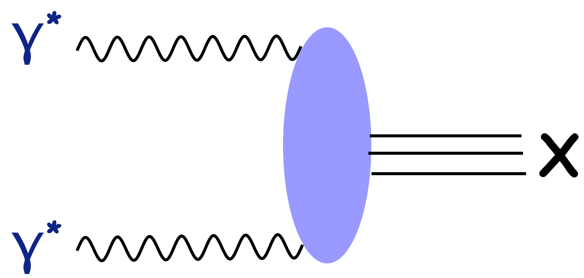
$M_{++,++},\ M_{+-,+-},\ M_{++,--},$ **T**

$M_{00,00},\ M_{+0,+0},\ M_{0+,0+},\ M_{++,00},\ M_{0+,-0}$ **T and L**

sum rules for LbL scattering (II)

Unitarity: link to $\gamma^* \gamma^* \to$ X cross sections

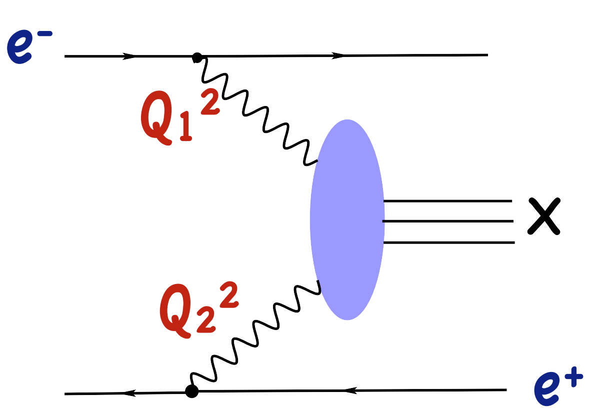
$$W_{\lambda_1'\lambda_2',\lambda_1\lambda_2} \equiv \mathrm{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}$$



$$\begin{aligned}
W_{++,++} + W_{+-,+-} &\equiv 2\sqrt{X}\,(\sigma_0 + \sigma_2) = 2\sqrt{X}\,(\sigma_\parallel + \sigma_\perp) \equiv 4\sqrt{X}\,\sigma_{TT},\\
W_{++,++} - W_{+-,+-} &\equiv 2\sqrt{X}\,(\sigma_0 - \sigma_2) \equiv 4\sqrt{X}\,\tau^a_{TT},\\
W_{++,--} &\equiv 2\sqrt{X}\,(\sigma_\parallel - \sigma_\perp) \equiv 2\sqrt{X}\,\tau_{TT},\\
W_{00,00} &\equiv 2\sqrt{X}\,\sigma_{LL},\\
W_{+0,+0} &\equiv 2\sqrt{X}\,\sigma_{TL},\\
W_{0+,0+} &\equiv 2\sqrt{X}\,\sigma_{LT},\\
W_{++,00} + W_{0+,-0} &\equiv 4\sqrt{X}\,\tau_{TL},\\
W_{++,00} - W_{0+,-0} &\equiv 4\sqrt{X}\,\tau^a_{TL}.
\end{aligned}$$

$$X \equiv \nu^2 - Q_1^2 Q_2^2$$

Experiment: e⁻ e⁺ → e⁻ e⁺ X cross sections

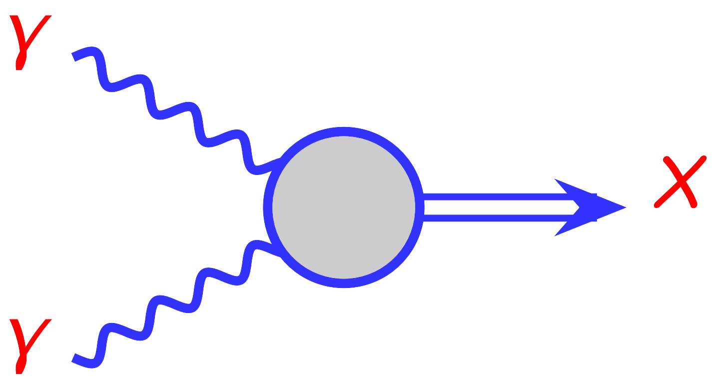


$$\begin{aligned}
d\sigma = &\frac{\alpha^2}{16\pi^4 Q_1^2 Q_2^2}\,\frac{2\sqrt{X}}{s(1-4m^2/s)} \cdot \frac{d^3\vec{p}_1{}'}{E_1'} \cdot \frac{d^3\vec{p}_2{}'}{E_2'}\\
&\times \Big\{ 4\,\rho_1^{++}\,\rho_2^{++}\,\sigma_{TT} + \rho_1^{00}\,\rho_2^{00}\,\sigma_{LL} + 2\,\rho_1^{++}\,\rho_2^{00}\,\sigma_{TL} + 2\,\rho_1^{00}\,\rho_2^{++}\,\sigma_{LT}\\
&+2\,\left(\rho_1^{++}-1\right)\,\left(\rho_2^{++}-1\right)\,\left(\cos 2\tilde{\phi}\right)\,\tau_{TT} + 8\left[\frac{\left(\rho_1^{00}+1\right)\,\left(\rho_2^{00}+1\right)}{\left(\rho_1^{++}-1\right)\,\left(\rho_2^{++}-1\right)}\right]^{1/2}\,\left(\cos\tilde{\phi}\right)\,\tau_{TL}\\
&+ h_1 h_2\, 4\,\left[\left(\rho_1^{00}+1\right)\,\left(\rho_2^{00}+1\right)\right]^{1/2}\,\tau^a_{TT} + h_1 h_2\, 8\,\left[\left(\rho_1^{++}-1\right)\,\left(\rho_2^{++}-1\right)\right]^{1/2}\,\left(\cos\tilde{\phi}\right)\,\tau^a_{TL} \Big\}
\end{aligned}$$

lepton beam polarization

ρ's, ϕ : kinematical quantities

sum rules for LbL scattering (III)

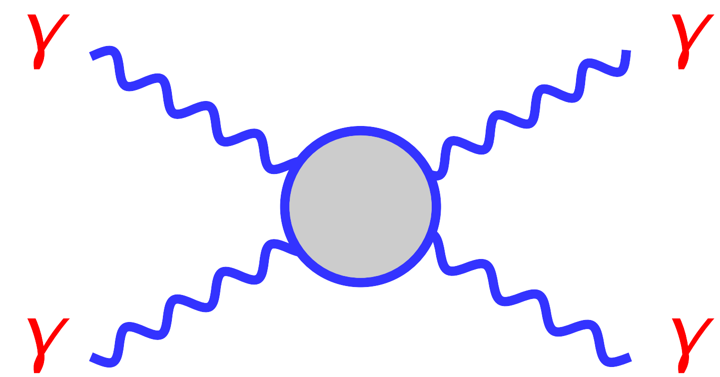


micro-causality

unitarity

dispersion theory

LETs



imaginary part of the amplitude - photon-photon fusion into leptons and hadrons:

real part of the amplitude - low-energy structure of the elastic LbL scattering:

Sum rules

$$\mathrm{Im} f^{(-)}(s) = -\tfrac{s}{8}\left[\sigma_2(s) - \sigma_0(s)\right]$$

$$\mathrm{Im} f^{(+)}(s) = -\tfrac{s}{8}\left[\sigma_{tot}(s)\right]$$

$$\mathcal{L}^{(8)} = c_1 (F_{\mu\nu}F^{\mu\nu})^2 + c_2 (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$$

Euler, Heisenberg (1936)

$$\int_{s_0}^{\infty} \frac{ds}{s}\left[\sigma_2(s) - \sigma_0(s)\right] = 0$$

$$c_1 \pm c_2 = \frac{1}{8\pi}\int_{s_0}^{\infty} \frac{ds}{s^2}\left[\sigma_{\parallel}(s) \pm \sigma_{\perp}(s)\right]$$

sum rules for LbL scattering (IV)

3 superconvergent relations:

Pascalutsa, Pauk, Vdh (2012)

helicity difference sum rule

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q_1^2)} [\sigma_0 - \sigma_2]_{Q_2^2=0}$$

for $Q^2 = 0$: GDH sum rule

Gerasimov, Moulin (1975), Brodsky, Schmidt (1995)

sum rules involving longitudinal photons

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q_1^2)^2} \left[\sigma_{\parallel} + \sigma_{LT} + \frac{(s+Q_1^2)}{Q_1 Q_2} \tau_{TL}^{a}\right]_{Q_2^2=0}$$

$$0 = \int_{s_0}^{\infty} ds \left[\frac{\tau_{TL}}{Q_1 Q_2}\right]_{Q_2^2=0}$$

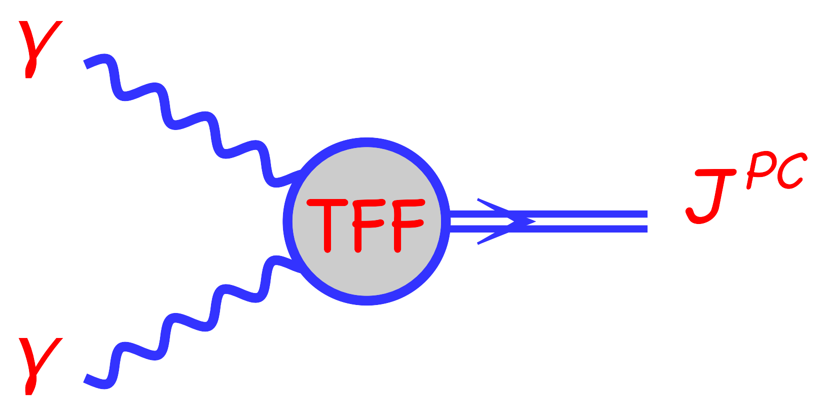
SRs involving LbL low-energy constants:

$$c_1 \pm c_2 = \frac{1}{8\pi} \int_{s_0}^{\infty} \frac{ds}{s^2} \left[\sigma_{\parallel}(s) \pm \sigma_{\perp}(s)\right]$$

+ 6 new LECs at next order

➡ sum rules have been tested in perturbative QFT both at tree-level and 1-loop level

single meson production in $\gamma\gamma$ collisions (I)



- two-photon state: produced meson has C=+1

- both photons are real: J=1 final state is forbidden (Landau-Yang theorem);

the main contribution comes from

J=0: 0^{-+} (pseudoscalar) and 0^{++} (scalar)

and J=2: 2^{++} (tensor)

– the SRs hold separately for channels of given intrinsic quantum numbers: isoscalar and isovector mesons, $c\bar{c}$ states

– input for the absorptive part of the SRs: γγ-hadrons response functions, can be expressed in terms of γγ→M transition form factors

$$\sigma_\Lambda^{\gamma\gamma\to M}(s) \approx (2J+1)16\pi^2 \frac{\Gamma_{\gamma\gamma}}{m_M}\delta(s-m_M^2)$$

meson contribution to the cross-section in the narrow-resonance approximation

$$\Gamma_{\gamma\gamma}(\mathcal{P}) = \frac{\pi\alpha^2}{4} m^3 \, |F_{\mathcal{M}\gamma^*\gamma^*}(0,0)|^2$$

two-photons decay rate for the meson

single meson production in $\gamma\gamma$ collisions (II)

the I=0 channel

	$\int \frac{ds}{s}(\sigma_2 - \sigma_0)$ [nb]	c_1 [10^{-4}GeV^{-4}]	c_2 [10^{-4}GeV^{-4}]
η	-191 ± 10	0	0.65 ± 0.03
η'	-300 ± 10	0	0.33 ± 0.01
$f_0(980)$	-19 ± 5	0.020 ± 0.005	0
$f_0'(1370)$	-91 ± 36	0.049 ± 0.019	0
$f_2(1270)$	449 ± 52	0.141 ± 0.016	0.141 ± 0.016
$f_2'(1525)$	7 ± 1	0.002 ± 0.000	0.002 ± 0.000
$f_2(1565)$	56 ± 11	0.012 ± 0.002	0.012 ± 0.002
Sum	-89 ± 66	0.22 ± 0.03	1.14 ± 0.04

dominant contribution to c_2 comes from η, η' and $f_2(1270)$

dominant contribution to c_1 comes from $f_2(1270)$

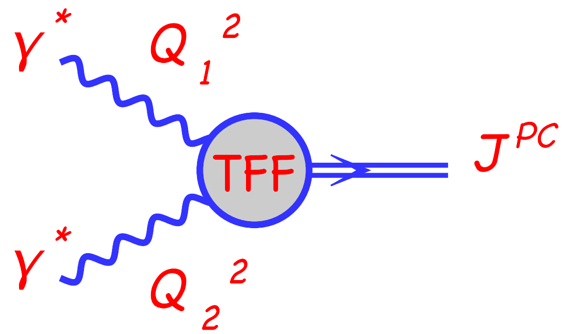
the I=1 channel

	$\int \frac{ds}{s}(\sigma_2 - \sigma_0)$ [nb]	c_1 [10^{-4} GeV^{-4}]	c_2 [10^{-4} GeV^{-4}]
π^0	-195 ± 13	0	10.94 ± 0.70
$a_0(980)$	-20 ± 8	0.021 ± 0.007	0
$a_2(1320)$	134 ± 8	0.039 ± 0.002	0.039 ± 0.002
$a_2(1700)$	18 ± 3	0.003 ± 0.001	0.003 ± 0.001
Sum	-63 ± 17	0.06 ± 0.01	10.98 ± 0.70

dominant contribution to c_2 comes from π^0

Pascalutsa, Pauk, Vdh (2012)

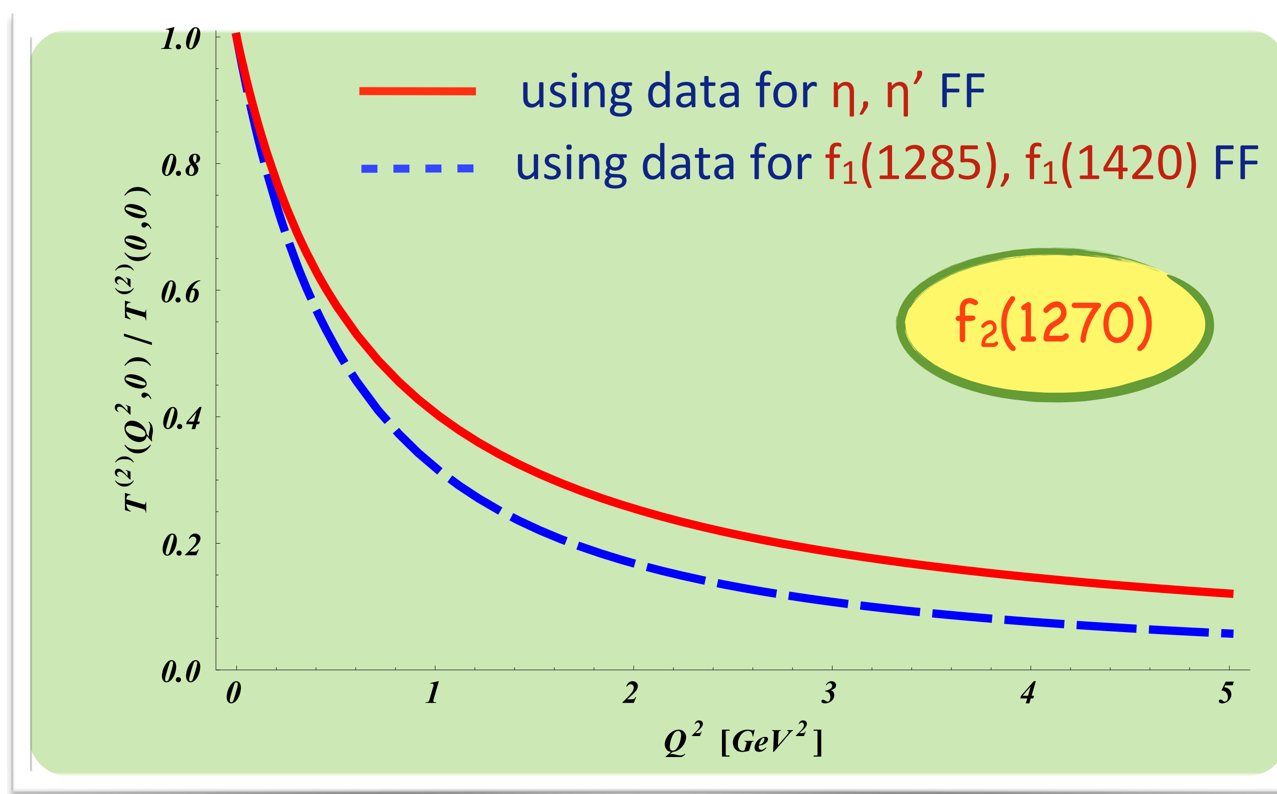
single meson production in $\gamma\gamma$ collisions (III)



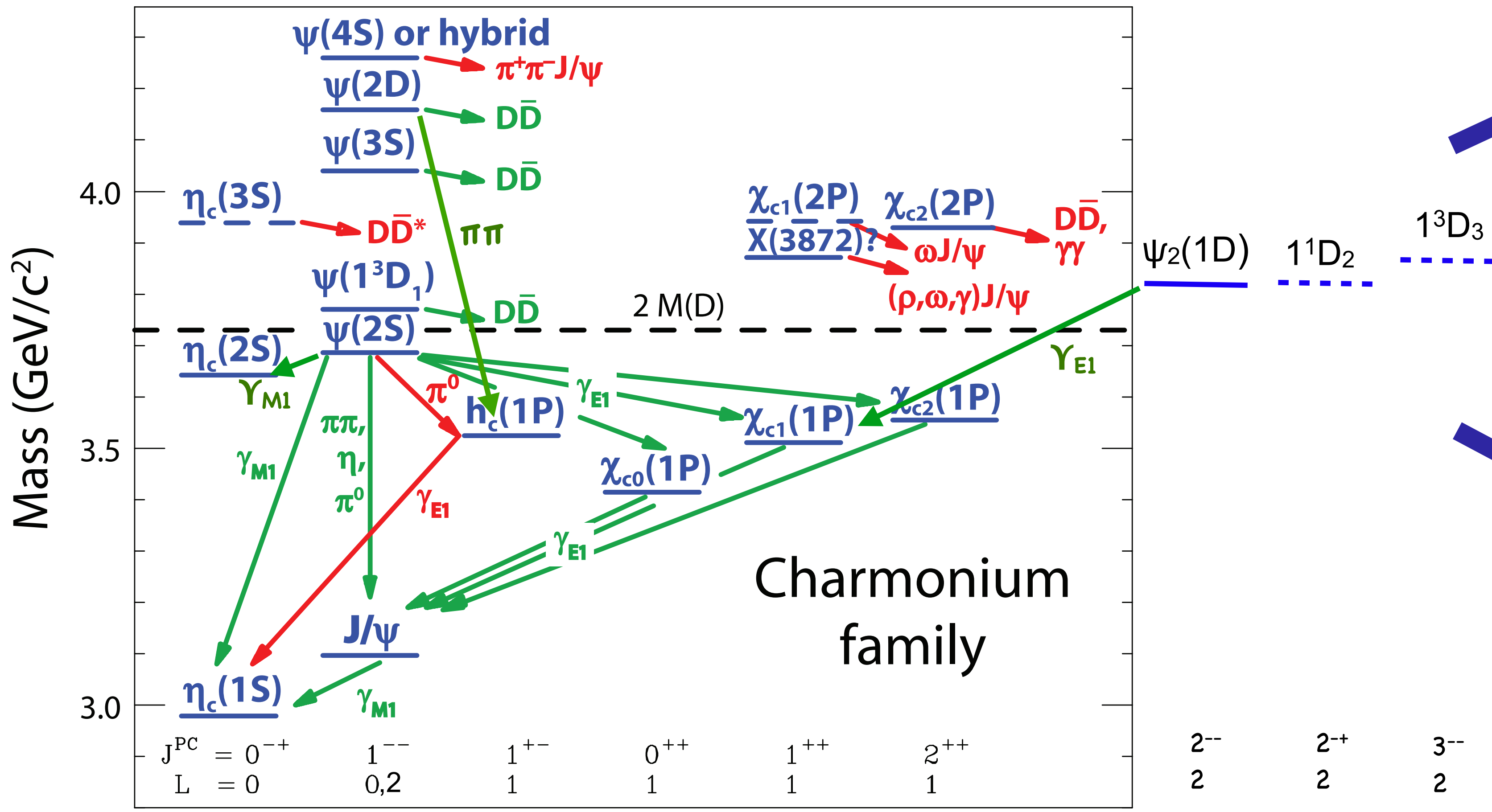
- one photon is virtual Q_1^2, second is quasi-real $Q_2^2 \simeq 0$:
- axial-vector mesons 1^{++} are allowed
- $f_1(1285)$, $f_1(1420)$ transition FFs constrained from LEP (L3) data

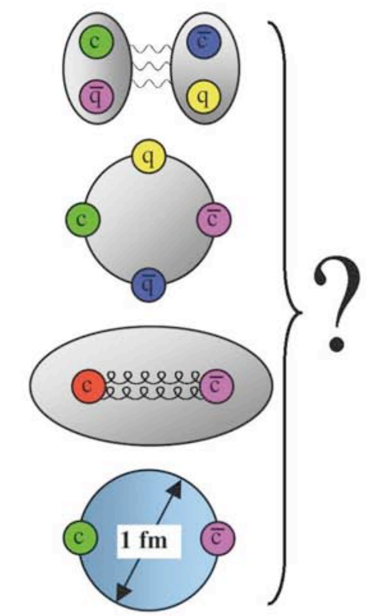
	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int \frac{ds}{s^2}\sigma_\parallel(s)$ [nb / GeV^2]	$\int ds \left[\frac{1}{s}\frac{\tau^a_{TL}}{Q_1 Q_2}\right]_{Q_i^2=0}$ [nb / GeV^2]	$\int ds \left[\frac{1}{s^2}\sigma_\parallel + \frac{1}{s}\frac{\tau^a_{TL}}{Q_1 Q_2}\right]_{Q_i^2=0}$ [nb / GeV^2]
$f_1(1285)$	1281.8 ± 0.6	3.5 ± 0.8	0	-93 ± 21	-93 ± 21
$f_1(1420)$	1426.4 ± 0.9	3.2 ± 0.9	0	-50 ± 14	-50 ± 14
$f_0(980)$	980 ± 10	0.29 ± 0.07	20 ± 5	0	20 ± 5
$f_0'(1370)$	$1200 - 1500$	3.8 ± 1.5	48 ± 19	0	48 ± 19
$f_2(1270)$	1275.1 ± 1.2	3.03 ± 0.35	138 ± 16	$\gtrsim 0$	138 ± 16
$f_2'(1525)$	1525 ± 5	0.081 ± 0.009	1.5 ± 0.2	$\gtrsim 0$	1.5 ± 0.2
$f_2(1565)$	1562 ± 13	0.70 ± 0.14	12 ± 2	$\gtrsim 0$	12 ± 2
Sum					76 ± 36

- sum rules allow to constrain so far unmeasured contributions, e.g. $\gamma^* \gamma^* \to$ tensor mesons



sum rules for charmonium states (I)





narrow states:
well
understood
$c\bar{c}$

sum rules for charmonium states (II)

sum rules evaluated for
$c\bar{c}$ states

$$0 = \int_{s_0}^{\infty} ds \, \frac{[\sigma_2 - \sigma_0](s)}{s}$$

	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int \frac{ds}{s} (\sigma_2 - \sigma_0)$ [nb]
0^{-+} $\eta_c(1S)$	2983.6 ± 1.2	5.06 ± 0.53	-11.9 ± 1.2
0^{++} $\chi_{c0}(1P)$	3414.75 ± 0.31	2.34 ± 0.27	-3.6 ± 0.4
2^{++} $\chi_{c2}(1P)$	3556.20 ± 0.09	0.53 ± 0.06	3.6 ± 0.4
Sum $c\bar{c}$ bound states			-11.9 ± 1.3
duality estimate continuum ($\sqrt{s} \geq 2m_D$)			14.9 ± 1.0
$c\bar{c}$ bound states + continuum			3.0 ± 2.3

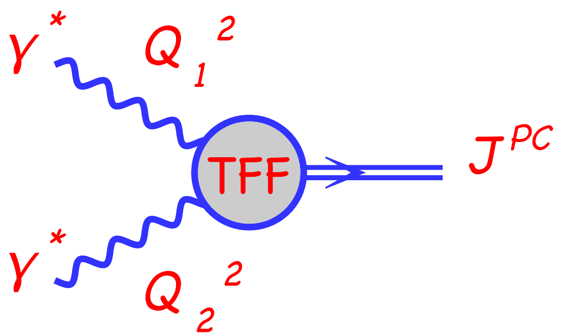
duality estimate for continuum contribution, above $D\bar{D}$ threshold

$$\int_{s_D}^{\infty} ds \, \frac{1}{s} \, [\sigma_2 - \sigma_0] \, (\gamma\gamma \to X) \approx \int_{s_D}^{\infty} ds \, \frac{1}{s} \, [\sigma_2 - \sigma_0] \, (\gamma\gamma \to c\bar{c})$$

interplay between hidden charm mesons ($c\bar{c}$ states) and production of charmed mesons

Pauk, Pascalutsa, Vdh (2012)

sum rules for charmonium states (III)



one photon virtual,
one quasi-real
also axial vector
states contribute

$$0 = \int_{s_0}^{\infty} ds \left[\frac{\sigma_{\parallel}}{s^2} + \frac{1}{s} \frac{\tau^a_{TL}}{Q_1 Q_2} \right]_{Q_1^2 = Q_2^2 = 0}$$

	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	$\int ds \left[\frac{1}{s^2}\sigma_{\parallel} + \frac{1}{s}\frac{\tau^a_{TL}}{Q_1 Q_2} \right]_{Q_i^2=0}$ [nb / GeV^2]
$\chi_{c0}(1P)$	3414.75 ± 0.31	2.34 ± 0.27	0.31 ± 0.04
$\chi_{c2}(1P)$	3556.20 ± 0.09	0.53 ± 0.06	$\gtrsim 0.14 \pm 0.02$
$\chi_{c1}(1P)$	3510.66 ± 0.07	-	$(-145 \pm 7) \cdot \left(\tilde{\Gamma}_{\gamma\gamma}/\Gamma\right)$
duality estimate continuum $(\sqrt{s} \geq 2m_D)$			-0.067 ± 0.005

saturating sum rule by **χ_{c1} (1P)**
allows a **prediction**:

$$\tilde{\Gamma}_{\gamma\gamma}(\mathcal{A}) \equiv \lim_{Q_1^2 \to 0} \frac{m_A^2}{Q_1^2} \frac{1}{2} \Gamma\left(\mathcal{A} \to \gamma_L^* \gamma_T\right)$$

$$\boxed{\tilde{\Gamma}_{\gamma\gamma}\left(\chi_{c1}\right) = (2.2 \pm 0.4)\,\text{keV}}$$

compare with

$$\tilde{\Gamma}_{\gamma\gamma}\left(f_1(1285)\right) = (3.5 \pm 0.8)\,\text{keV}$$

L3 Coll.

Summary and outlook

- **new theoretical tools** for $\gamma^* \gamma^* \to X$

  - sum rules, dispersive frameworks for transition FFs**:**
  allow to include experimental constraints

  - new evaluation of **heavier meson contributions:** $a_\mu = (6.6 \sim 4.4) \pm 2.9 \times 10^{-11}$

- **new dispersion relation** frameworks for $\mathbf{a_\mu}$:
  -> require close collaboration with experiment (spacelike, timelike, meson decays)

- Outcome of Mainz workshop:
  draft of roadmap for a data driven approach in HLbL

- **goal: realistic error estimate on a_μ / reduce to 2 x 10^{-10} (20 % of HLbL)**

Section 1

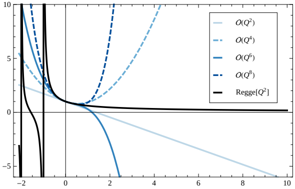
Appendix

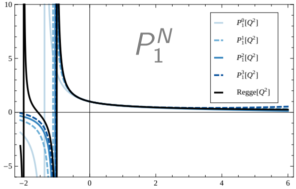
Padé Approximants: Convergence properties

Convergence known for meromorphic (large-N_c) and Stieltjes (DR)

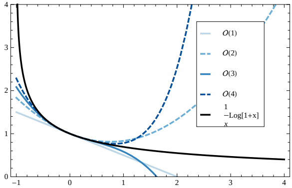
for the last $\lim_{N\to\infty} P^N_{N+1}(x) \leq f(x) \leq P^N_N(x)$

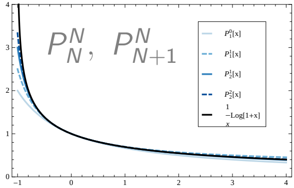
$$— \quad F_{P\gamma^*\gamma}(Q^2) = \frac{F_{P\gamma\gamma}(0)}{Q^2} \frac{a}{\psi^{(1)}\left(\frac{M^2}{a}\right)} \left(\psi^{(0)}\left(\frac{M^2+Q^2}{a}\right) - \psi^{(0)}\left(\frac{M^2}{a}\right)\right) \quad —$$





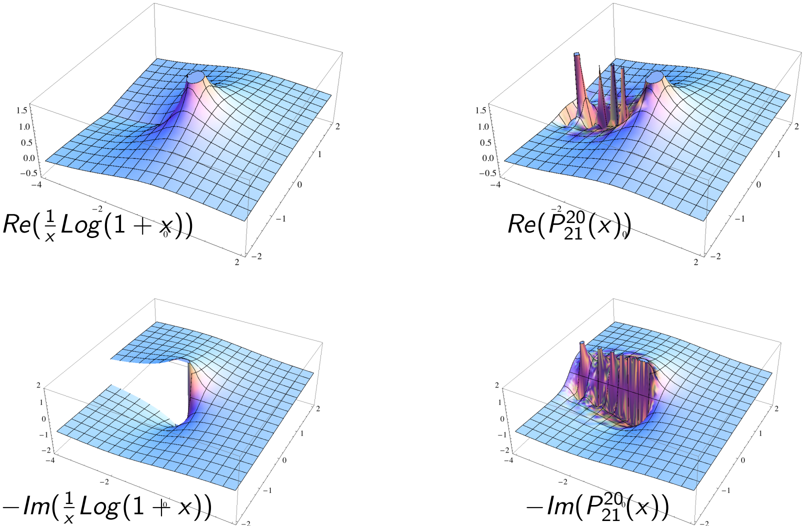
TAYLOR —— $f(x) = \frac{1}{x} Log(1+x)$ —— PADÉ





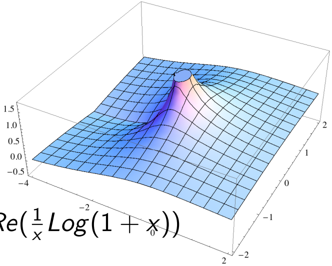
Padé Approximants: Convergence properties

II. Stieljes functions: $(1/x)ln(1+x)$

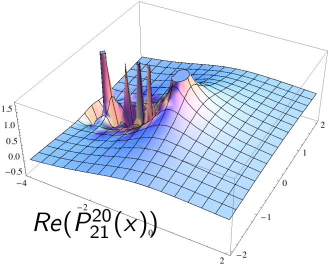


Padé Approximants: Convergence properties

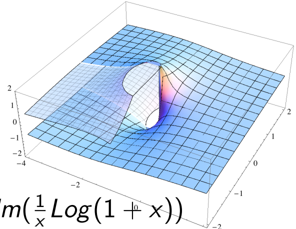
II. Stieljes functions: $(1/x)ln(1+x)$



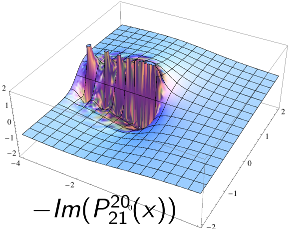
$Re(\frac{1}{x}Log(1+x))$



$Re(P^{20}_{21}(x))$



$-Im(\frac{1}{x}Log(1+x))$

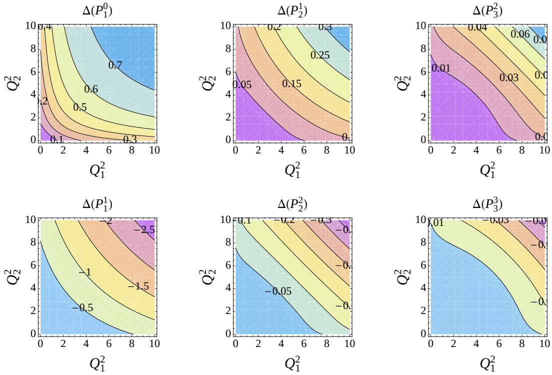


$-Im(P^{20}_{21}(x))$

Our proposal: Bivariate Padé Approximants

Lets revisit the Regge Model

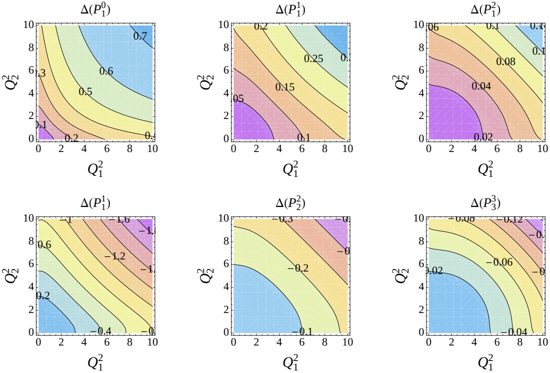
$$F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = \frac{F_{P\gamma\gamma}(0,0)}{Q_1^2 - Q_2^2} \frac{a}{\psi^{(1)}\left(\frac{M^2}{a}\right)} \left(\psi^{(0)}\left(\frac{M^2+Q_1^2}{a}\right) - \psi^{(0)}\left(\frac{M^2+Q_2^2}{a}\right)\right)$$



Obeys $P^N_{N+1}(x,y) \leq f(x,y) \leq P^N_N(x,y)$ (Stieltjes)

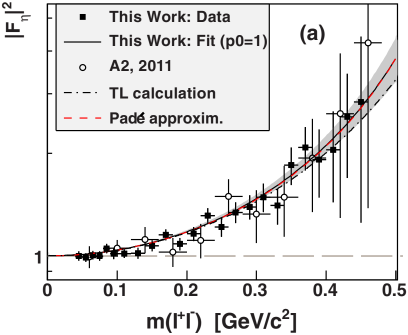
Our proposal: Bivariate Padé Approximants

A bigger challenge: cuts $F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) = F_{P\gamma\gamma}(0,0)\frac{M^2}{Q_1^2 - Q_2^2} ln\left(\frac{M^2+Q_1^2}{M^2+Q_2^2}\right)$



Obeys $P_{N+1}^N(x,y) \leq f(x,y) \leq P_N^N(x,y)$ (Stieltjes)

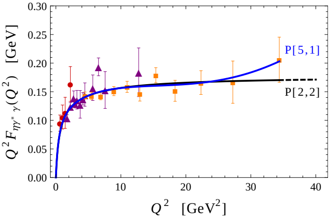
Dalitz decays: $\eta \to \gamma\bar{\ell}\ell$



Compare to A2 Coll. results in Mainz [Phys.Rev. C89 (2014) 044608]
The results are excellent $\to$ reasonable to use them in our fit

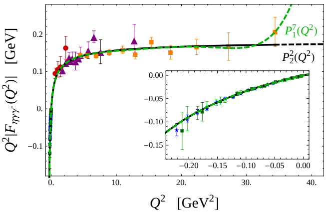
R. Escribano, P. Masjuan, P. Sanchez, In preparation

Dalitz decays: $\eta \to \gamma\bar{\ell}\ell$



Previous Results

$b_\eta = 0.60(6)(3)(m_\eta)^{-2}$
$c_\eta = 0.37(10)(7)(m_\eta)^{-4}$
$d_\eta = -$
Asymptotics $= 0.160(24)$ GeV



Updated Results

$b_\eta = 0.576(11)(1)(m_\eta)^{-2}$
$c_\eta = 0.339(15)(2)(m_\eta)^{-4}$
$d_\eta = 0.200(14)(10)(m_\eta)^{-6}$
Asymptotics $= 0.177(15)$ GeV

R. Escribano, P. Masjuan, P. Sanchez, In preparation

Toy Model for $P \to \bar{\ell}\ell$: Unitarity Analtiticity & Cuts

$$F_{P\gamma^*\gamma^*}(q_1^2, q_2^2) = F_{P\gamma\gamma^*}(q_1^2) \times F_{P\gamma\gamma^*}(q_2^2)$$

$$F_{P\gamma\gamma^*}(q^2) = c_{P\rho} G_\rho(q^2) + c_{P\omega} G_\omega(q^2) c_{P\phi} G_\phi(q^2)$$

Based on Dumm, Pich, Portoles PRD62 and Dumm, Roig, EPJC73

$$G_\rho(s) = \frac{M_V^2}{M_\rho^2 - s + \frac{sM_\rho^2}{96\pi^2 F_\pi^2}\left(ln\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} - \sigma(s)^3 ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right)\right)}$$

For narrow resonances

$$G_{\omega,\phi} = \frac{M_{\omega,\phi} + M_{\omega,\phi}\Gamma_{\omega,\phi}\sqrt{s_{th}/M_{\omega,\phi}}}{M_{\omega,\phi} - s + M_{\omega,\phi}\Gamma_{\omega,\phi}\sqrt{(s_{th}-s)/M_{\omega,\phi}}}$$

Toy Model for $P \to \bar{\ell}\ell$: Unitarity Analtiticity & Cuts

Integration is easy through Cauchy's integral Formula

$$G(q^2) = \frac{1}{\pi}\int_{s_{th}}^{\infty} dM^2 \frac{Im\left[G(M^2)\right]}{M^2 - q^2 - i\epsilon}$$

Then our loop integral can be solved through standard procedures

$$\mathcal{A} = \frac{1}{\pi^2}\int_{s_{th}}^{\infty}\int_{s_{th}}^{\infty} dM_1^2 dM_2^2 Im\left[G(M_1^2)\right] Im\left[G(M_2^2)\right] \times \\ \times \int_{Loop} d^4k[...] \frac{1}{k^2 - M_1^2 + i\epsilon}\frac{1}{(q-k)^2 - M_2^2 + i\epsilon}$$