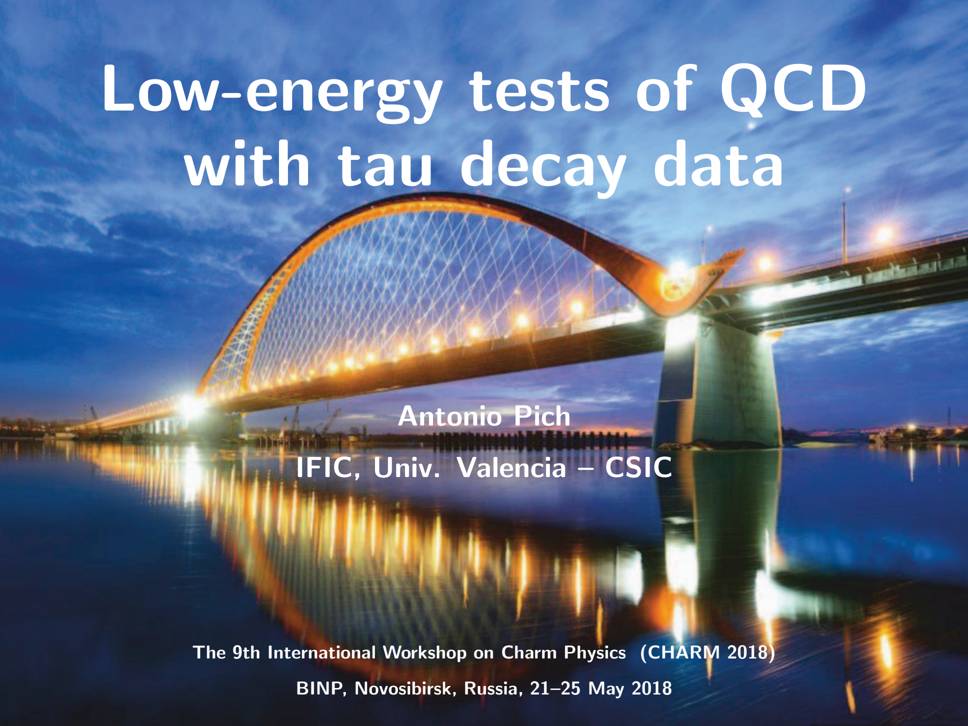

Low-energy tests of QCD with tau decay data

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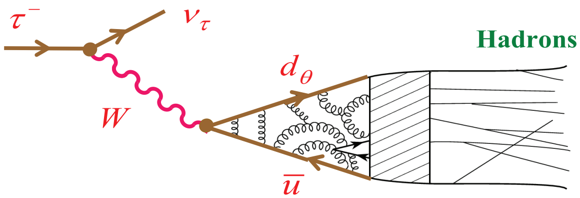
The 9th International Workshop on Charm Physics (CHARM 2018)

BINP, Novosibirsk, Russia, 21–25 May 2018

Hadronic τ Decay



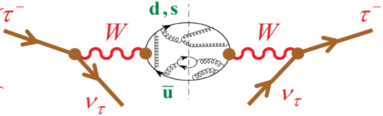
$$d_\theta = V_{ud}\, d + V_{us}\, s$$

Only lepton massive enough to decay into hadrons

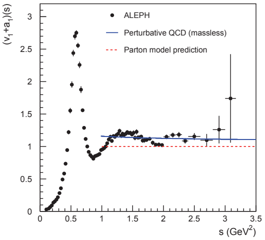
$$R_\tau \equiv \frac{\Gamma\left(\tau^- \to \nu_\tau + \mathrm{Hadrons}\right)}{\Gamma\left(\tau^- \to \nu_\tau\, e^-\, \overline{\nu}_e\right)} \approx N_C \qquad ; \qquad R_\tau = \frac{1 - B_e - B_\mu}{B_e} = 3.637 \pm 0.011$$

$$R_\tau = \frac{1}{B_e^{\mathrm{univ}}} - 1.97256 = 3.6407 \pm 0.0072 \qquad ; \qquad R_\tau = \frac{\mathrm{Br}(\tau^- \to \nu_\tau + \mathrm{Had.})}{B_e} = 3.6349 \pm 0.0082$$

τ Hadronic Width: R_τ

$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\Big\{$  $\Big\}$

Davier et al, 1312.1501



$$\Pi^{(J)}(s) \;\equiv\; |V_{ud}|^2 \left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) \;+\; |V_{us}|^2\; \Pi^{(J)}_{us,V+A}(s)$$

$$v_1 \;=\; 2\pi\; \text{Im}\Pi^{(1)}_{ud,V}(s) \quad , \quad a_1 \;=\; 2\pi\; \text{Im}\Pi^{(1)}_{ud,A}(s)$$

$$R_\tau \;=\; \frac{\Gamma[\tau^- \to \nu_\tau \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \;=\; R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$$=\; 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im}\,\Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2}\,\text{Im}\,\Pi^{(0)}(s)\right]$$

Theoretical Framework

$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \mathrm{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

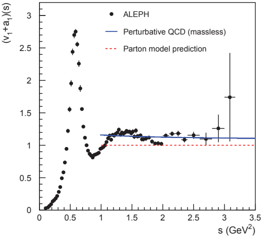
$$= 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right]$$

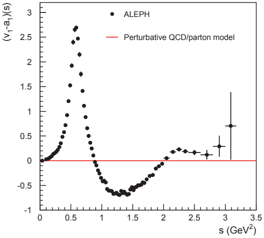
$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) + |V_{us}|^2\, \Pi^{(J)}_{us,V+A}(s)$$

Davier et al, 1312.1501

$$v_1 = 2\pi\, \mathrm{Im}\Pi^{(1)}_{ud,V}(s)$$

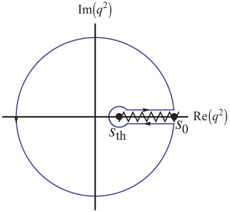
$$a_1 = 2\pi\, \mathrm{Im}\Pi^{(1)}_{ud,A}(s)$$





$$i\int d^4x\, e^{iqx}\, \langle 0|\, T\left[\mathcal{J}^{\mu}_{ij}(x)\mathcal{J}^{\nu\,\dagger}_{ij}(0)\right]\, |0\rangle \;=\; \left(-g^{\mu\nu}q^2 + q^\mu q^\nu\right)\Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu\, \Pi^{(0)}_{ij,\mathcal{J}}(q^2)$$

$$A^{\omega}_{\mathcal{J}}(s_0) \;\equiv\; \int_{s_{\rm th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) \;=\; -\frac{1}{2i}\oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi^{(J)}_{\mathcal{J}}(s)$$



$$\Pi^{(J)}_{\mathcal{J}}(s) \;\approx\; \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} \;=\; \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$\omega(s) = s^N \quad \rightarrow \quad \mathcal{O}^{(J)}_{2N+2,\mathcal{J}}$$

$$\delta_{\rm DV} \;\equiv\; \frac{1}{2\pi i}\oint_{|s|=s_0} ds\, \omega(s)\, \left\{\Pi^{(J)}_{\mathcal{J}}(s) - \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE}\right\} \;=\; \frac{1}{\pi}\int_{s_0}^{\infty} ds\, \omega(s)\, \mathrm{Im}\left[\Pi^{(J)}_{\mathcal{J}}(s) - \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE}\right]$$

Add residues whenever there are poles within the integration contour

Left-Right Correlator

$$\Pi(s) \;\equiv\; \Pi^{(0+1)}_{ud,LR}(s) \;\equiv\; \Pi^{(0+1)}_{ud,V}(s) - \Pi^{(0+1)}_{ud,A}(s) \;=\; \frac{2f_\pi^2}{s-m_\pi^2} + \overline{\Pi}(s)$$

- **Pure non-perturbative quantity:** Ideal to test the OPE
- **Known short-distance constraints:** $\Pi(s)^{\mathrm{OPE}}_{m_q=0} \;=\; \sum_{D\geq 6} \frac{\mathcal{O}_D}{(-s)^{D/2}}$

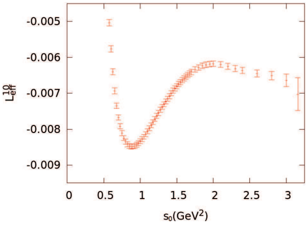
  1. **1st WSR:** $$\int_{s_{\mathrm{th}}}^{\infty} ds\; \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \;=\; 2f_\pi^2$$
  2. **2nd WSR:** $$\int_{s_{\mathrm{th}}}^{\infty} ds\; s\; \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \;=\; 2f_\pi^2 m_\pi^2$$
  3. **πSR:** $$\int_{s_{\mathrm{th}}}^{\infty} ds\; s\;\log\left(\frac{s}{\Lambda^2}\right)\frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)\Big|_{m_q=0} \;=\; \left(m^2_{\pi^0}-m^2_{\pi^+}\right)_{\mathrm{em}}\;\frac{8\pi}{3\alpha}\;f_\pi^2\Big|_{m_q=0}$$

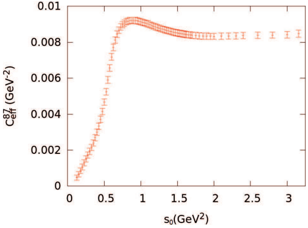
- **χPT:** $\overline{\Pi}(s) \;=\; -8\,L_{10} + \chi\mathrm{logs} + s\;[16\,C_{87} + \chi\mathrm{logs}] + \mathcal{O}(s^2)$

Non-Pinched & Pinched Weights

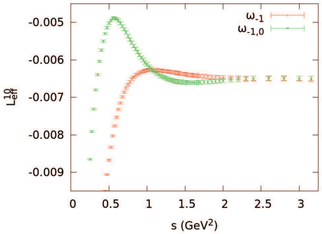
González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

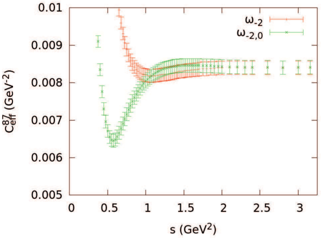
$$L_{10}^{\mathrm{eff}} = -\frac{1}{8}\int_{s_{\mathrm{th}}}^{s_0} ds\; s^{-1}\,\frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \qquad\qquad C_{87}^{\mathrm{eff}} = \frac{1}{16}\int_{s_{\mathrm{th}}}^{s_0} ds\; s^{-2}\,\frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)$$





$$\omega_{-1,0} = s^{-1}\,(1 - s/s_0)\quad,\quad \omega_{-1} = s^{-1}\,(1 - s/s_0)^2\quad,\quad \omega_{-2,0} = s^{-2}\,(1 - s^2/s_0^2)\quad,\quad \omega_{-2} = s^{-2}\,(1 - s/s_0)^2(1 + 2s/s_0)$$





Playing with Duality Violations

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

- **Ad-hoc Ansatz:** $\rho(s > s_z) \,=\, \kappa\, e^{-\gamma s}\, \sin\{\beta(s - s_z)\}$ (Cata et al)

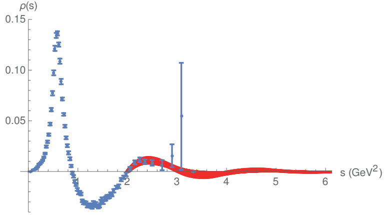
- **$3 \cdot 10^6$ randomly generated tuples $(\kappa, \gamma, \beta, s_z)$**

- **Select "Acceptable" Spectral Functions, satisfying:**

  1. Consistent with ALEPH at 90% C.L., above $s = 1.7\ \mathrm{GeV}^2$
  2. Fulfill 1st and 2nd WSRs, and πSR

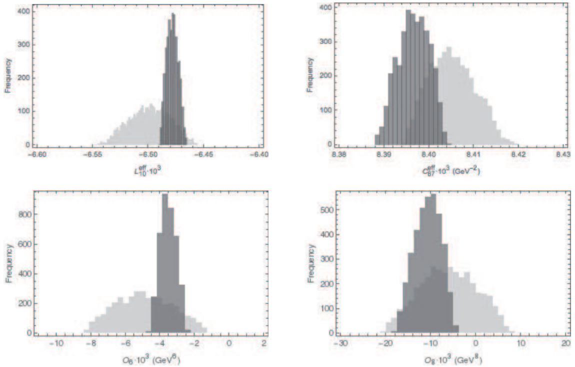
3716 tuples selected

(red band)



Statistical Distributions of $L_{10}^{\rm eff}$, $C_{87}^{\rm eff}$, $\mathcal{O}_6$ and $\mathcal{O}_8$

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112



Non-pinched

& pinched

weights

$$L_{10}^{\rm eff} = \left(-6.477\,^{+\,0.004}_{-\,0.006} \pm 0.05\right)\cdot 10^{-3} = \left(-6.48 \pm 0.05\right)\cdot 10^{-3}$$

$$C_{87}^{\rm eff} = \left(8.399\,^{+\,0.002}_{-\,0.005} \pm 0.18\right)\cdot 10^{-3}\ {\rm GeV}^{-2} = \left(8.40 \pm 0.18\right)\cdot 10^{-3}\ {\rm GeV}^{-2}$$

$$\mathcal{O}_6 = \left(-3.6\,^{+\,0.5}_{-\,0.4} \pm 0.5\right)\cdot 10^{-3}\ {\rm GeV}^{6} = \left(-3.6\,^{+\,0.7}_{-\,0.6}\right)\cdot 10^{-3}\ {\rm GeV}^{6}$$

$$\mathcal{O}_8 = \left(-1.0 \pm 0.3 \pm 0.2\right)\cdot 10^{-2}\ {\rm GeV}^{8} = \left(-1.0 \pm 0.4\right)\cdot 10^{-2}\ {\rm GeV}^{8}$$

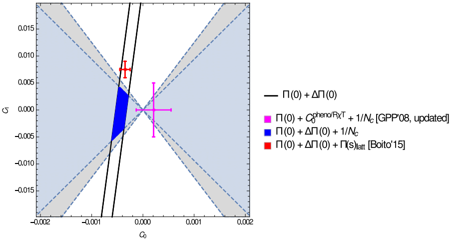
• χPT Parameters:

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

$$L_{10}^{\rm eff} = L_{10}^r - 0.00126 + \mathcal{O}(p^6)$$

$$L_{10}^{\rm eff} = 1.53\, L_{10}^r + 0.263\, L_9^r - 0.00179 - \frac{1}{8}\left(\mathcal{C}_0^r + \mathcal{C}_1^r\right) + \mathcal{O}(p^8)$$

$$C_{87}^{\rm eff} = C_{87}^r + 0.296\, L_9^r + 0.00155 + \mathcal{O}(p^8)$$



- $\mathcal{O}(p^4)$ analysis: $L_{10}^r(M_\rho) = -(5.22 \pm 0.05) \cdot 10^{-3}$
- $\mathcal{O}(p^6)$ analysis: $L_{10}^r(M_\rho) = -(4.1 \pm 0.4) \cdot 10^{-3}$

  $C_{87}^r(M_\rho) = (5.10 \pm 0.22) \cdot 10^{-3}\ \mathrm{GeV}^{-2}$

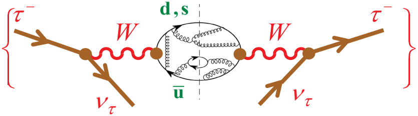
• $\varepsilon'_K/\varepsilon_K$:

$\mathcal{O}_6$ → $\langle(\pi\pi)_{I=2}|Q_8|K^0\rangle$ → e.m. penguin contribution

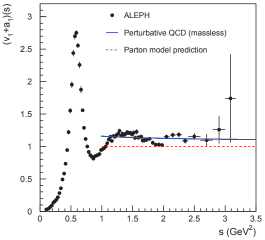
Good agreement with large-N_C approximation (work in progress)

τ Hadronic Width: R_τ

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\left\{ \right\}$$



Davier et al, 1312.1501



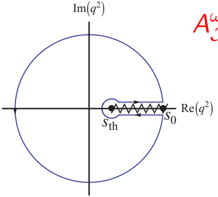
$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) + |V_{us}|^2\, \Pi^{(J)}_{us,V+A}(s)$$

$$v_1 = 2\pi\, \text{Im}\Pi^{(1)}_{ud,V}(s) \quad , \quad a_1 = 2\pi\, \text{Im}\Pi^{(1)}_{ud,A}(s)$$

$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$$= 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im}\,\Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2}\,\text{Im}\,\Pi^{(0)}(s)\right]$$

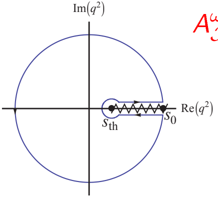
$$i \int d^4x\, e^{iqx}\, \langle 0|\, T\left[\mathcal{J}^{\mu}_{ij}(x)\, \mathcal{J}^{\nu\,\dagger}_{ij}(0)\right]\, |0\rangle \;=\; \left(-g^{\mu\nu}\, q^2 + q^{\mu}\, q^{\nu}\right)\, \Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^{\mu}\, q^{\nu}\; \Pi^{(0)}_{ij,\mathcal{J}}(q^2)$$



$$A^{\omega}_{\mathcal{J}}(s_0) \;\equiv\; \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi^{(J)}_{\mathcal{J}}(s)$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \;\approx\; \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} \;=\; \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$i \int d^4x \, e^{iqx} \, \langle 0| \, T \left[\mathcal{J}_{ij}^{\mu}(x) \, \mathcal{J}_{ij}^{\nu\,\dagger}(0) \right] |0\rangle \; = \; (-g^{\mu\nu} q^2 + q^{\mu} q^{\nu}) \; \Pi_{ij,\mathcal{J}}^{(1)}(q^2) + q^{\mu} q^{\nu} \; \Pi_{ij,\mathcal{J}}^{(0)}(q^2)$$



$$A_{\mathcal{J}}^{\omega}(s_0) \;\equiv\; \int_{s_{th}}^{s_0} \frac{ds}{s_0} \, \omega(s) \; \mathrm{Im}\,\Pi_{\mathcal{J}}^{(J)}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0} \, \omega(s) \; \Pi_{\mathcal{J}}^{(J)}(s)$$

$$\Pi_{\mathcal{J}}^{(J)}(s) \;\approx\; \Pi_{\mathcal{J}}^{(J)}(s)^{\mathrm{OPE}} \;=\; \sum_D \frac{\mathcal{O}_{D,\mathcal{J}}^{(J)}}{(-s)^{D/2}}$$

$$\mathbf{R}_{\tau} \;=\; \mathbf{N_C \, S_{EW}} \, (1 + \delta_{\mathbf{P}} + \delta_{\mathbf{NP}})$$ Braaten-Narison-Pich '92

$$= \; 6\pi i \oint_{|x|=1} (1-x)^2 \left[(1+2x) \, \Pi^{(0+1)}(m_\tau^2 x) - 2x \, \Pi^{(0)}(m_\tau^2 x) \right]$$

$\delta_{\mathbf{P}} = a_\tau + 5.20 \, a_\tau^2 + 26 \, a_\tau^3 + 127 \, a_\tau^4 + \cdots \approx 20\%$ Baikov-Chetyrkin-Kühn '08

$a_\tau \equiv \alpha_s(m_\tau^2)/\pi$, $\mathbf{S_{EW}} = 1.0201 \, (3)$ Marciano-Sirlin, Braaten-Li, Erler

$\delta_{\mathbf{NP}} = -0.0064 \pm 0.0013$ (Fitted from data) Davier et al '14

Perturbative Contribution $(m_q = 0)$

$$a_\tau \equiv \frac{\alpha_s(m_\tau^2)}{\pi}$$

$$-s\,\frac{d}{ds}\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n\, a_s(-s)^n \quad \Longrightarrow \quad \delta_{\mathrm{P}} = \underbrace{\sum_{n=1} K_n\, A^{(n)}(\alpha_s)}_{\mathrm{CIPT}} = \underbrace{\sum_{n=1} r_n\, a_\tau^n}_{\mathrm{FOPT}}$$

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\left(1-2x+2x^3-x^4\right)\left(\frac{\alpha_s(-m_\tau^2 x)}{\pi}\right)^n = a_\tau^n + \cdots$$

Perturbative Contribution $(m_q = 0)$

$a_\tau \equiv \frac{\alpha_s(m_\tau^2)}{\pi}$

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1) The dominant corrections come from the contour integration

Large running of α_s along the circle $s = m_\tau^2\, e^{i\phi}$, $\phi \in [-\pi, \pi]$

n	1	2	3	4	5
K_n	1	1.6398	6.37101	49.0757	?
r_n	1	5.2023	26.3659	127.079	$307.78 + K_5$
$r_n - K_n$	0	3.5625	19.9949	78.0029	307.78

Baikov-Chetyrkin-Kühn '08

Le Diberder-Pich '92

Perturbative Contribution $(m_q = 0)$

$a_\tau \equiv \frac{\alpha_s(m_\tau^2)}{\pi}$

$$-s\,\frac{d}{ds}\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n\, a_s(-s)^n \quad \Rightarrow \quad \delta_{\rm P} = \underbrace{\sum_{n=1} K_n\, A^{(n)}(\alpha_s)}_{\rm CIPT} = \underbrace{\sum_{n=1} r_n\, a_\tau^n}_{\rm FOPT}$$

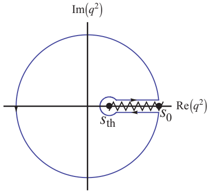
$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\left(1-2x+2x^3-x^4\right)\left(\frac{\alpha_s(-m_\tau^2 x)}{\pi}\right)^n = a_\tau^n + \cdots$$

2) CIPT gives rise to a well-behaved perturbative series

$a_\tau = 0.11$	$A^{(1)}(\alpha_s)$	$A^{(2)}(\alpha_s)$	$A^{(3)}(\alpha_s)$	$A^{(4)}(\alpha_s)$	$\delta_{\rm P}$
$\beta_{n>1} = 0$	0.14828	0.01925	0.00225	0.00024	0.20578
$\beta_{n>2} = 0$	0.15103	0.01905	0.00209	0.00020	0.20537
$\beta_{n>3} = 0$	0.15093	0.01882	0.00202	0.00019	0.20389
$\beta_{n>4} = 0$	0.15058	0.01865	0.00198	0.00018	0.20273
$\beta_{n>5} = 0$	0.15041	0.01859	0.00197	0.00018	0.20232
$\mathcal{O}(a_\tau^4)$ FOPT	0.16115	0.02431	0.00290	0.00015	0.22665

FOPT overestimates $\delta_{\rm P}$ by 11%

Non-Perturbative Contribution

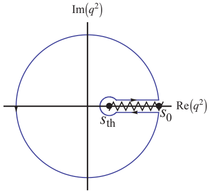


$$A^{\omega}_{\mathcal{J}}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\; \Pi^{(J)}_{\mathcal{J}}(s)$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \approx \Pi^{(J)}_{\mathcal{J}}(s)^{\mathrm{OPE}} = \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$A^{\omega,\mathrm{NP}}_{\mathcal{J}}(s_0) = \pi \sum_D a_{-1,D}\, \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{s_0^{D/2}} \qquad , \qquad \omega(-s_0 x) = \sum_n a_{n,D}\; x^{n+D/2}$$

Non-Perturbative Contribution



$$A^{\omega}_{\mathcal{J}}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi^{(J)}_{\mathcal{J}}(s)$$

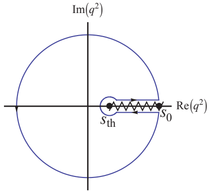
$$\Pi^{(J)}_{\mathcal{J}}(s) \approx \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} = \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$A^{\omega,\rm NP}_{\mathcal{J}}(s_0) = \pi \sum_D a_{-1,D}\, \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{s_0^{D/2}} \qquad , \qquad \omega(-s_0 x) = \sum_n a_{n,D}\; x^{n+D/2}$$

- **Strong power suppression at $\mathbf{s_0 = m_\tau^2}$:** $\quad \sim (\Lambda_{\rm QCD}/m_\tau)^D \quad , \quad D \geq 4$

$$\mathcal{O}_{4,V/A} \approx 4\pi^2 \left\{ \tfrac{1}{12\pi} \langle \alpha_s GG\rangle + (m_u + m_d)\,\langle \bar q q\rangle \right\} \approx [(6.7 \pm 3.2) - 0.6]\cdot 10^{-3} \times m_\tau^4$$

Non-Perturbative Contribution



$$A^\omega_{\mathcal{J}}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\,\Pi^{(J)}_{\mathcal{J}}(s) = \frac{i}{2}\oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi^{(J)}_{\mathcal{J}}(s)$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \approx \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} = \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$A^{\omega,\rm NP}_{\mathcal{J}}(s_0) = \pi \sum_D a_{-1,D}\, \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{s_0^{D/2}} \quad , \quad \omega(-s_0 x) = \sum_n a_{n,D}\, x^{n+D/2}$$

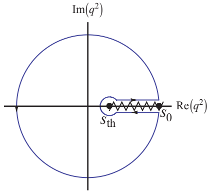
- **Strong power suppression at $\mathbf{s_0 = m_\tau^2}$:** $\sim (\Lambda_{\rm QCD}/m_\tau)^D \quad , \quad D \geq 4$

$$\mathcal{O}_{4,V/A} \approx 4\pi^2 \left\{ \tfrac{1}{12\pi}\langle \alpha_s GG\rangle + (m_u + m_d)\langle \bar{q}q\rangle \right\} \approx [(6.7 \pm 3.2) - 0.6]\cdot 10^{-3} \times m_\tau^4$$

- **$\mathbf{R_\tau}$:** $\boldsymbol{\omega(x) = 1 - 3x^2 + 2x^3} \;\longrightarrow\; \boldsymbol{\delta_{\rm NP} = -3\,\frac{\mathcal{O}_{6,V+A}}{m_\tau^6} - 2\,\frac{\mathcal{O}_{8,V+A}}{m_\tau^8}}$

Additional chiral suppression in $|\mathcal{O}_{6,V+A}| < |\mathcal{O}_{6,V-A}| \approx (1.1 \pm 0.2)\cdot 10^{-4} \times m_\tau^6$

Non-Perturbative Contribution



$$A^{\omega}_{\mathcal{J}}(s_0) \equiv \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\, \Pi^{(J)}_{\mathcal{J}}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\; \Pi^{(J)}_{\mathcal{J}}(s)$$

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- **Strong power suppression at $\mathbf{s_0 = m_\tau^2}$:** $\sim (\Lambda_{\mathrm{QCD}}/m_\tau)^D \quad , \quad D \geq 4$

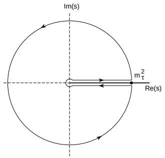
$$\mathcal{O}_{4,V/A} \approx 4\pi^2 \left\{ \tfrac{1}{12\pi} \langle \alpha_s GG \rangle + (m_u + m_d)\, \langle \bar{q} q \rangle \right\} \approx [(6.7 \pm 3.2) - 0.6] \cdot 10^{-3} \times m_\tau^4$$

- **$\mathbf{R_\tau}$:** $\boldsymbol{\omega(x) = 1 - 3x^2 + 2x^3}$ ⟶ $\boldsymbol{\delta_{\mathrm{NP}} = -3\, \frac{\mathcal{O}_{6,V+A}}{m_\tau^6} - 2\, \frac{\mathcal{O}_{8,V+A}}{m_\tau^8}}$

Additional chiral suppression in $|\mathcal{O}_{6,V+A}| < |\mathcal{O}_{6,V-A}| \approx (1.1 \pm 0.2) \cdot 10^{-4} \times m_\tau^6$

- **Sensitivity to $\mathbf{\mathcal{O}_D}$ with different $\boldsymbol{\omega(x)}$** ⟶ **Measure $\boldsymbol{\delta_{\mathrm{NP}}}$**

R_τ suitable for a precise α_s determination



$$R_\tau = 6\pi i \oint_{|x|=1} (1-x)^2 \left[(1+2x)\, \Pi^{(0+1)}(m_\tau^2 x) - 2x\, \Pi^{(0)}(m_\tau^2 x) \right]$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \approx \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} = \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

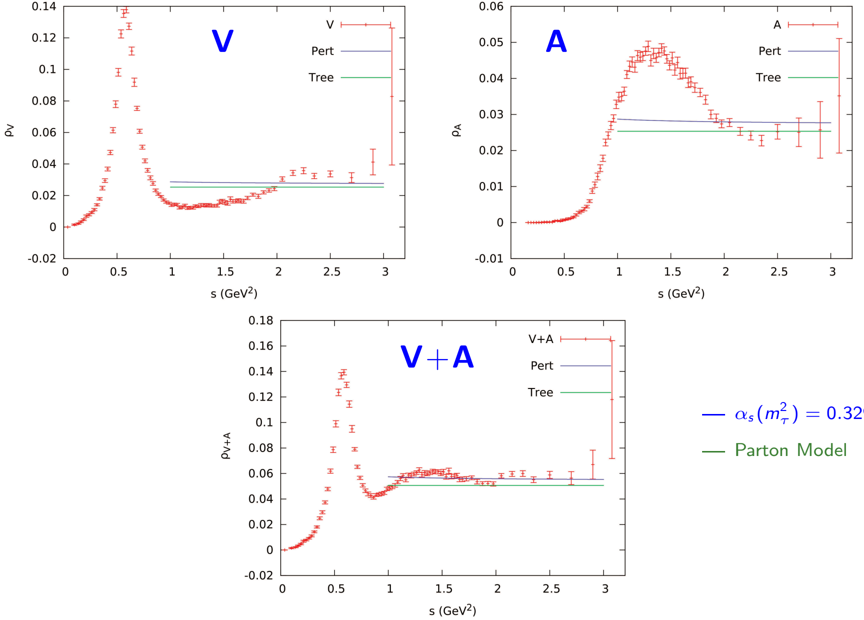
- **Known to $\mathcal{O}(\alpha_s^4)$. Sizeable $\delta_P \sim 20\%$. Strong sensitivity to α_s**
- **m_τ large enough to safely use the OPE. Flat $V+A$ distribution**
- **OPE only valid away from real axis: $(1-x)^2$ pinched at $s = m_\tau^2$**
- $m_{u,d} = 0$ ➡ $s\,\Pi^{(0)} = 0$ ➡ $R_{\tau,V+A} = 6\pi i \oint_{|x|=1} (1-3x^2+2x^3)\, \Pi^{(0+1)}_{ud,V+A}(m_\tau^2 x)$

  ➡ **$\delta_{NP} \sim 1/m_\tau^6$ Strong suppression of non-perturbative effects**
- **$D=6$ OPE contributions have opposite sign for V & A. Cancellation**
- **δ_{NP} can be determined from data: $\delta_{NP} = -0.0064 \pm 0.0013$** Davier et al

R_τ ➡ $\boxed{\alpha_s(m_\tau^2) = 0.331 \pm 0.013}$ Pich 2014

ALEPH Spectral Functions

Davier et al. 2014



— $\alpha_s(m_\tau^2) = 0.329$

— Parton Model

New analysis of ALEPH data

Rodríguez-Sánchez, Pich, arXiv:1605.06830

Method (V + A)	$\alpha_s(m_\tau^2)$		
	CIPT	FOPT	Average
ALEPH moments [1]	$0.339\,^{+0.019}_{-0.017}$	$0.319\,^{+0.017}_{-0.015}$	$0.329\,^{+0.020}_{-0.018}$
Mod. ALEPH moments [2]	$0.338\,^{+0.014}_{-0.012}$	$0.319\,^{+0.013}_{-0.010}$	$0.329\,^{+0.016}_{-0.014}$
$A^{(2,m)}$ moments [3]	$0.336\,^{+0.018}_{-0.016}$	$0.317\,^{+0.015}_{-0.013}$	$0.326\,^{+0.018}_{-0.016}$
s_0 dependence [4]	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
Borel transform [5]	$0.328\,^{+0.014}_{-0.013}$	$0.318\,^{+0.015}_{-0.012}$	$0.323\,^{+0.015}_{-0.013}$
Combined value	$\mathbf{0.335 \pm 0.013}$	$\mathbf{0.320 \pm 0.012}$	$\mathbf{0.328 \pm 0.013}$

→ $$\alpha_s(M_Z^2) = 0.1197 \pm 0.0015$$

1) $\omega_{kl}(x) = (1+2x)\,(1-x)^{2+k}\,x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

2) $\tilde{\omega}_{kl}(x) = (1-x)^{2+k}\,x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

3) $\omega^{(2,m)}(x) = (1-x)^2 \sum_{k=0}^{m} (k+1)\,x^k = 1 - (m+2)\,x^{m+1} + (m+1)x^{m+2}$, $\qquad 1 \le m \le 5$

4) $\omega^{(2,m)}(x)$ $\qquad 0 \le m \le 2$, 1 single moment in each fit

5) $\omega_a^{(1,m)}(x) = (1 - x^{m+1})\,\mathrm{e}^{-ax}$ $\qquad 0 \le m \le 6$

α_s determination with ALEPH-like fit

Rodríguez-Sánchez, A.P.

$\omega_{kl}(x) \; = \; (1-x)^{2+k}\, x^l\, (1+2x)$, $x = s/m_\tau^2$, $(k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

Channel	$\alpha_s(m_\tau^2)/\pi$	$\langle \frac{\alpha_s}{\pi} GG \rangle$ (10^{-3} GeV^4)	$\mathcal{O}_6$ (10^{-3} GeV^6)	$\mathcal{O}_8$ 10^{-3} GeV^8)
V (FOPT)	$0.328\,^{+0.013}_{-0.007}$	$8\,^{+7}_{-14}$	$-3.2\,^{+0.8}_{-0.5}$	$5.0\,^{+0.4}_{-0.7}$
V (CIPT)	$0.352\,^{+0.013}_{-0.011}$	$-8\,^{+7}_{-7}$	$-3.5\,^{+0.3}_{-0.3}$	$4.9\,^{+0.4}_{-0.5}$
A (FOPT)	$0.304\,^{+0.010}_{-0.007}$	$-15\,^{+5}_{-8}$	$4.4\,^{+0.5}_{-0.4}$	$-5.8\,^{+0.3}_{-0.4}$
A (CIPT)	$0.320\,^{+0.011}_{-0.010}$	$-25\,^{+5}_{-5}$	$4.3\,^{+0.2}_{-0.2}$	$-5.8\,^{+0.3}_{-0.3}$
V+A (FOPT)	$0.319\,^{+0.010}_{-0.006}$	$-3\,^{+6}_{-11}$	$1.3\,^{+1.4}_{-0.8}$	$-0.8\,^{+0.4}_{-0.7}$
V+A (CIPT)	$0.339\,^{+0.011}_{-0.009}$	$-16\,^{+5}_{-5}$	$0.9\,^{+0.3}_{-0.4}$	$-1.0\,^{+0.5}_{-0.7}$

- **High sensitivity to α_s. Bad sensitivity to power corrections**
- **Cancellation in $\mathcal{O}_{6,V+A}$ confirmed. $V+A$ more reliable**
- $K_5 = 275 \pm 400,\ \mu^2 = (0.5, 2)\, m_\tau^2$
- Best values taken from $V+A$. Errors increased with sensitivity to $\mathcal{O}_{10}$

$$\alpha_s(m_\tau^2)^{\rm CIPT} = 0.339\,^{+0.019}_{-0.017}$$

$$\alpha_s(m_\tau^2)^{\rm FOPT} = 0.319\,^{+0.017}_{-0.015}$$

$\longrightarrow$

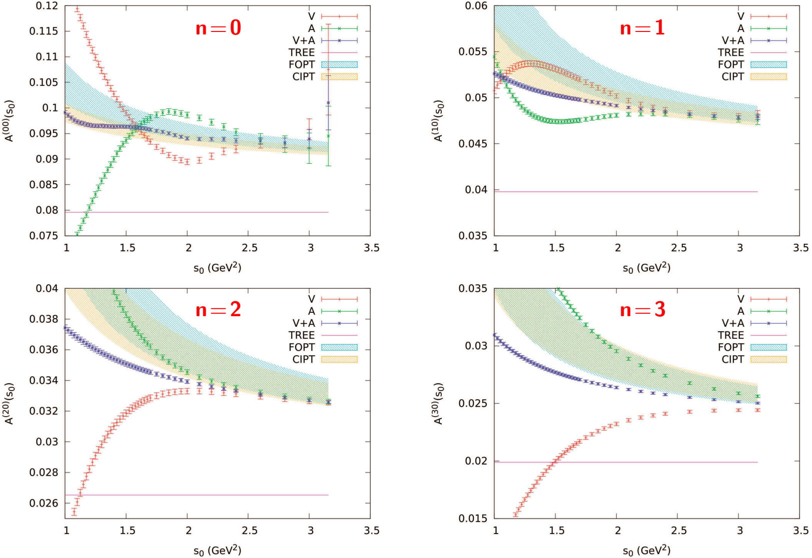
$$\boxed{\alpha_s(m_\tau^2) = 0.329\,^{+0.020}_{-0.018}}$$

Good agreement with Davier et al.: $\alpha_s(m_\tau^2) = 0.332 \pm 0.012$ (arXiv:1312.1501)

Experiment vs. (pinched) Perturbation Theory (only)

Rodríguez-Sánchez, A.P.

$\omega^{(n,0)}(s = s_0 x) \; = \; (1-x)^n$ ➡ $\mathcal{O}_{D\leq 2\,(n+1)}$, $\alpha_s(m_\tau^2) \; = \; 0.329\,^{+\,0.020}_{-\,0.018}$



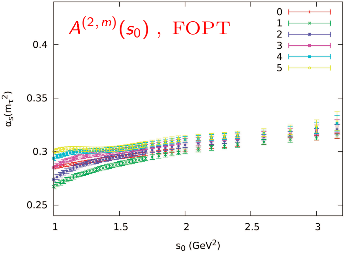
Non-Perturbative Contributions Neglected

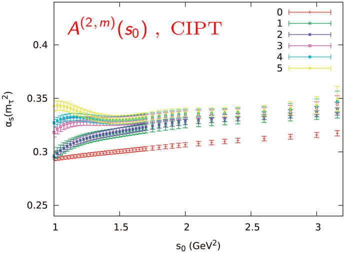
Rodríguez-Sánchez, A.P.

$$\omega^{(1,m)}(x) \;=\; 1 - x^{m+1} \;\;\rightarrow\;\; \mathcal{O}_{2m+4}$$

$$\omega^{(2,m)}(x) \;=\; (1-x)^2 \sum_{k=0}^{m} (k+1)\, x^k \;=\; 1 - (m+2)\, x^{m+1} + (m+1)\, x^{m+2} \;\;\rightarrow\;\; \mathcal{O}_{2m+4,2m+6}$$

Moment	$\alpha_s(m_\tau^2)$		Moment	$\alpha_s(m_\tau^2)$	
(n,m)	FOPT	CIPT	(n,m)	FOPT	CIPT
(1,0)	$0.315\,^{+0.012}_{-0.007}$	$0.327\,^{+0.012}_{-0.009}$	(2,0)	$0.311\,^{+0.015}_{-0.011}$	$0.314\,^{+0.013}_{-0.009}$
(1,1)	$0.319\,^{+0.010}_{-0.006}$	$0.340\,^{+0.011}_{-0.009}$	(2,1)	$0.311\,^{+0.011}_{-0.006}$	$0.333\,^{+0.009}_{-0.007}$
(1,2)	$0.322\,^{+0.010}_{-0.008}$	$0.343\,^{+0.012}_{-0.010}$	(2,2)	$0.316\,^{+0.010}_{-0.005}$	$0.336\,^{+0.011}_{-0.009}$
(1,3)	$0.324\,^{+0.011}_{-0.010}$	$0.345\,^{+0.013}_{-0.011}$	(2,3)	$0.318\,^{+0.010}_{-0.006}$	$0.339\,^{+0.011}_{-0.008}$
(1,4)	$0.326\,^{+0.011}_{-0.011}$	$0.347\,^{+0.013}_{-0.012}$	(2,4)	$0.319\,^{+0.009}_{-0.007}$	$0.340\,^{+0.011}_{-0.009}$
(1,5)	$0.327\,^{+0.015}_{-0.013}$	$0.348\,^{+0.014}_{-0.012}$	(2,5)	$0.320\,^{+0.010}_{-0.008}$	$0.341\,^{+0.011}_{-0.009}$





$V+A$

Exp. errors only

Non-Perturbative Contributions Neglected

Rodríguez-Sánchez, A.P.

$$\omega^{(1,m)}(x) = 1 - x^{m+1} \rightarrow \mathcal{O}_{2m+4}$$

$$\omega^{(2,m)}(x) = (1-x)^2 \sum_{k=0}^{m} (k+1)\, x^k = 1 - (m+2)\, x^{m+1} + (m+1)\, x^{m+2} \rightarrow \mathcal{O}_{2m+4,2m+6}$$

Moment (n,m)	$\alpha_s(m_\tau^2)$ FOPT	$\alpha_s(m_\tau^2)$ CIPT	Moment (n,m)	$\alpha_s(m_\tau^2)$ FOPT	$\alpha_s(m_\tau^2)$ CIPT
(1,0)	$0.315\,^{+0.012}_{-0.007}$	$0.327\,^{+0.012}_{-0.009}$	(2,0)	$0.311\,^{+0.015}_{-0.011}$	$0.314\,^{+0.013}_{-0.009}$
(1,1)	$0.319\,^{+0.010}_{-0.006}$	$0.340\,^{+0.011}_{-0.009}$	(2,1)	$0.311\,^{+0.011}_{-0.006}$	$0.333\,^{+0.009}_{-0.007}$
(1,2)	$0.322\,^{+0.010}_{-0.008}$	$0.343\,^{+0.012}_{-0.010}$	(2,2)	$0.316\,^{+0.010}_{-0.005}$	$0.336\,^{+0.011}_{-0.009}$
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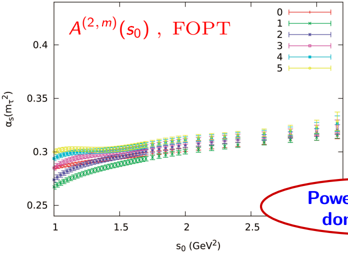
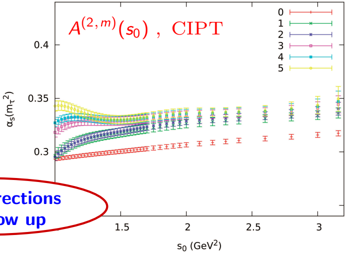
Amazing stability





Power corrections don't show up

V + A

Exp. errors only

Models of Duality Violation

$$\Delta A^{\omega}_{V/A}(s_0) \; = \; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s) \left\{ \Pi_{V/A}\,(s) - \Pi^{\mathrm{OPE}}_{V/A}(s) \right\} \; = \; -\pi \int_{s_0}^{\infty} \frac{ds}{s_0}\, \omega(s)\, \Delta\rho^{\mathrm{DV}}_{V/A}(s)$$

Models of Duality Violation

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Ansatz: $\Delta\rho^{\rm DV}_{V/A}(s) \; = \; s^{\lambda_{V/A}}\; e^{-(\delta_{V/A}+\gamma_{V/A}\, s)} \; \sin\left(\alpha_{V/A} + \beta_{V/A}\, s\right) \quad , \quad s > \hat{s}_0$

1) Boito et al.: $\lambda_{V/A} = 0 \quad , \quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad , \quad \omega(x) = 1$

- Fit s_0 dependence: ➡ $\{A^{(00)}(s_0), \rho(s_0+\Delta s_0), \cdots, \rho(s_0+(n-1)\Delta s_0)\}$
- Direct fit of the spectral function. **OPE not valid**

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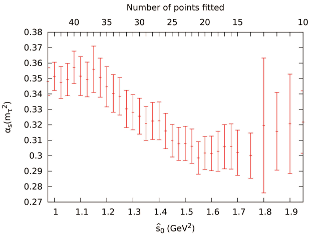

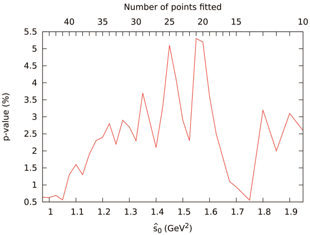

Rodríguez-Sánchez, A.P.

FOPT , V

(too large errors in A)

Bad quality fit **(Model dependence. Instabilities. Very low p-value)**

Models of Duality Violation

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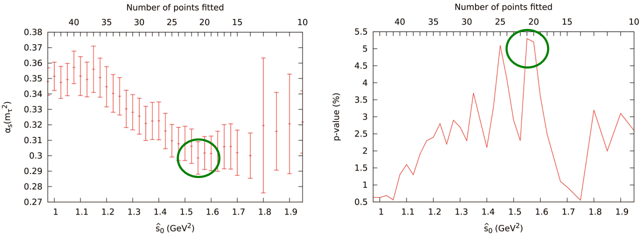


Rodríguez-Sánchez, A.P.

FOPT , V

(too large errors in A)

Boito et al. value

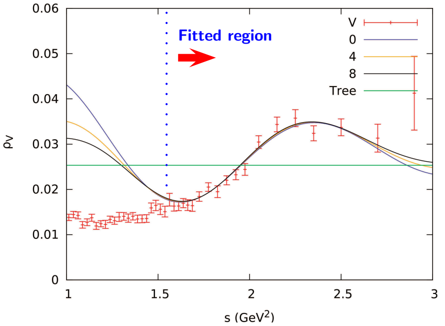
Bad quality fit **(Model dependence. Instabilities. Very low p-value)**

Ansatz: $\Delta\rho^{\mathrm{DV}}_{V/A}(s) \;=\; s^{\lambda_{V/A}}\, e^{-(\delta_{V/A}+\gamma_{V/A}\, s)}\, \sin\left(\alpha_{V/A}+\beta_{V/A}\, s\right) \quad , \quad s>\hat{s}_0$

2) $\lambda_V \;\geq\; 0$: $\quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad , \quad \omega(x)=1$

Rodríguez-Sánchez, A.P.

λ_V	$\alpha_s(m_\tau^2)^{\mathrm{FOPT}}$	δ_V	γ_V	α_V	β_V	p-value
0	0.298 ± 0.010	3.6 ± 0.5	0.6 ± 0.3	-2.3 ± 0.9	4.3 ± 0.5	5.3 %
1	0.300 ± 0.012	3.3 ± 0.5	1.1 ± 0.3	-2.2 ± 1.0	4.2 ± 0.5	5.7 %
2	0.302 ± 0.011	2.9 ± 0.5	1.6 ± 0.3	-2.2 ± 0.9	4.2 ± 0.5	6.0 %
4	0.306 ± 0.013	2.3 ± 0.5	2.6 ± 0.3	-1.9 ± 0.9	4.1 ± 0.5	6.6 %
8	0.314 ± 0.015	1.0 ± 0.5	4.6 ± 0.3	-1.5 ± 1.1	3.9 ± 0.6	7.7 %



- **Fitted α_s is model dependent**
- **$\lambda_V = 0$** (Boito) **gives the worse fit**
- **Fit quality & α_s increase with λ_V**

  ➔ **closer to data at $s < \hat{s}_0$**
- **$\Delta\hat{s}_0$ ➔ 3 times larger errors**

Not competitive & unreliable

Ansatz: $\Delta\rho^{\mathrm{DV}}_{V/A}(s) \; = \; s^{\lambda_{V/A}} \; e^{-(\delta_{V/A}+\gamma_{V/A}\, s)} \; \sin\left(\alpha_{V/A} + \beta_{V/A}\, s\right) \quad , \quad s > \hat{s}_0$

2) $\lambda_V \;\geq\; 0$: $\quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad , \quad \omega(x) = 1$

Rodríguez-Sánchez, A.P.

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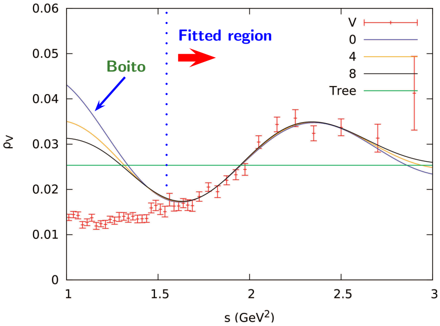
	4	0.306 ± 0.013	2.3 ± 0.5	2.6 ± 0.3	-1.9 ± 0.9	4.1 ± 0.5	6.6 %
	8	0.314 ± 0.015	1.0 ± 0.5	4.6 ± 0.3	-1.5 ± 1.1	3.9 ± 0.6	7.7 %



- **Fitted α_s is model dependent**
- **$\lambda_V = 0$ (Boito) gives the worse fit**
- **Fit quality & α_s increase with λ_V**

  ➡ **closer to data at $s \; < \; \hat{s}_0$**
- **$\Delta\hat{s}_0$ ➡ 3 times larger errors**

Not competitive & unreliable

New analysis of ALEPH data

Rodríguez-Sánchez, Pich, arXiv:1605.06830

Method (V + A)	$\alpha_s(m_\tau^2)$		
	CIPT	FOPT	Average
ALEPH moments [1]	$0.339\,^{+0.019}_{-0.017}$	$0.319\,^{+0.017}_{-0.015}$	$0.329\,^{+0.020}_{-0.018}$
Mod. ALEPH moments [2]	$0.338\,^{+0.014}_{-0.012}$	$0.319\,^{+0.013}_{-0.010}$	$0.329\,^{+0.016}_{-0.014}$
$A^{(2,m)}$ moments [3]	$0.336\,^{+0.018}_{-0.016}$	$0.317\,^{+0.015}_{-0.013}$	$0.326\,^{+0.018}_{-0.016}$
s_0 dependence [4]	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
Borel transform [5]	$0.328\,^{+0.014}_{-0.013}$	$0.318\,^{+0.015}_{-0.012}$	$0.323\,^{+0.015}_{-0.013}$
Combined value	$\mathbf{0.335 \pm 0.013}$	$\mathbf{0.320 \pm 0.012}$	$\mathbf{0.328 \pm 0.013}$

$$\longrightarrow \quad \alpha_s(M_Z^2) = 0.1197 \pm 0.0015$$

1) $\omega_{kl}(x) = (1+2x)\,(1-x)^{2+k}\,x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

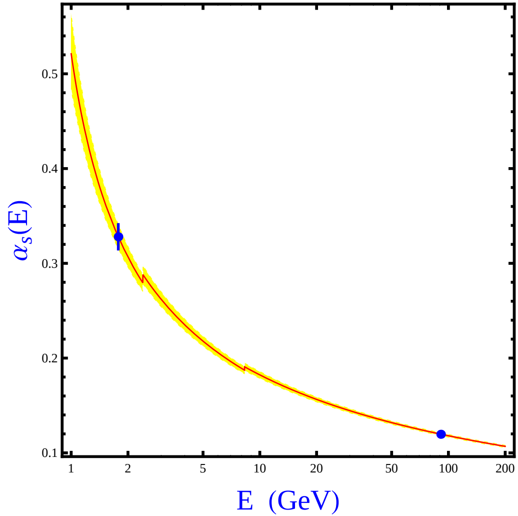
2) $\tilde{\omega}_{kl}(x) = (1-x)^{2+k}\,x^l$ $\qquad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

3) $\omega^{(2,m)}(x) = (1-x)^2 \sum_{k=0}^{m} (k+1)\,x^k = 1-(m+2)\,x^{m+1} + (m+1)x^{m+2}$, $\qquad 1 \le m \le 5$

4) $\omega^{(2,m)}(x)$ $\qquad 0 \le m \le 2$, 1 single moment in each fit

5) $\omega_a^{(1,m)}(x) = (1-x^{m+1})\,\mathrm{e}^{-ax}$ $\qquad 0 \le m \le 6$

α_s at N^{3}LO from τ and Z



$$\alpha_s(m_\tau^2) = 0.328 \pm 0.013$$

$$\alpha_s(M_Z^2) = 0.1197 \pm 0.0015$$

$$\alpha_s(M_Z^2)_{\text{Z width}} = 0.1196 \pm 0.0030$$

The most precise test of Asymptotic Freedom

$$\alpha_s^\tau(M_Z^2) - \alpha_s^Z(M_Z^2) = 0.0001 \pm 0.0015_\tau \pm 0.0030_Z$$

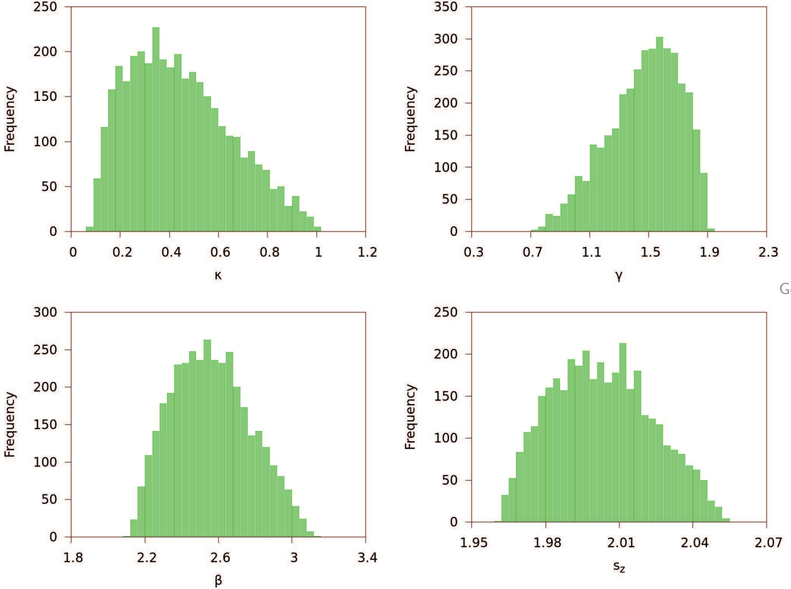
Backup

The 9th International Workshop on Charm Physics (CHARM 2018)

BINP, Novosibirsk, Russia, 21–25 May 2018

Statistical Distributions of Selected Tuples

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

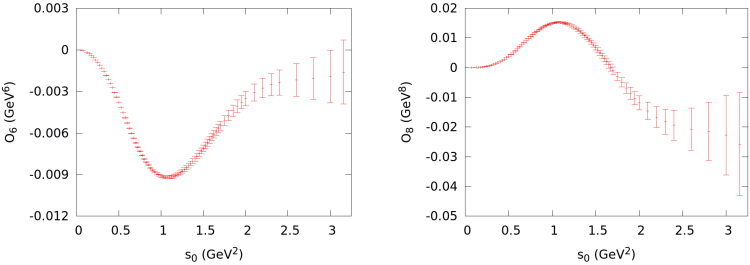


GeV units

$\mathcal{O}_{6,8}$ with Pinched Weights, ignoring DV

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

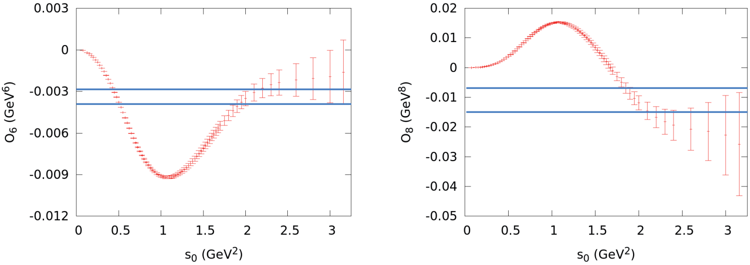
$$\int_{s_{\rm th}}^{s_0} ds\; \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)\;\; (s-s_0)^2\;\; \{1\,,\; (s+2s_0)\}$$



$\mathcal{O}_{6,8}$ with Pinched Weights, ignoring DV

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

$$\int_{s_{\rm th}}^{s_0} ds\ \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s)\ (s-s_0)^2\ \{1\,,\ (s+2s_0)\}$$



Result with DV fully taken into account

Pinched weights suppress very efficiently duality violation effects

Big reduction of errors mainly due to short-distance constraints which overcome the large data uncertainties at large s_0 values

Comparison with Previous Works

González-Alonso, Pich, Rodríguez-Sánchez, 1602.06112

$10^3 \cdot L_{10}^{\rm eff}$	$10^3 \cdot C_{87}^{\rm eff}$ (GeV^{-2})	$10^3 \cdot \mathcal{O}_6$ (GeV^6)	$10^2 \cdot \mathcal{O}_8$ (GeV^8)	Reference	Comments
-6.45 ± 0.06	–	-2.3 ± 0.6	-5.4 ± 3.3	BPDS'06	ALEPH'05 + DV=0
–	–	$-6.8^{+2.0}_{-0.8}$	$3.2^{+2.8}_{-9.2}$	ASS'08	ALEPH'05 + DV=0
-6.48 ± 0.06	8.18 ± 0.14	–	–	GPP'08	ALEPH'05 + DV=0
-6.44 ± 0.05	8.17 ± 0.12	-4.4 ± 0.8	-0.7 ± 0.5	GPP'10	ALEPH'05 + DV_{V-A}
-6.45 ± 0.09	8.47 ± 0.29	-6.6 ± 1.1	0.5 ± 0.5	Boito'12	OPAL'99 + $\mathrm{DV}_{V/A}$
-6.50 ± 0.10	–	-5.0 ± 0.7	-0.9 ± 0.5	DHSS'15	ALEPH'14 + DV=0
-6.45 ± 0.05	8.38 ± 0.18	-3.2 ± 0.9	-1.3 ± 0.6	Boito'15	ALEPH'14 + $\mathrm{DV}_{V/A}$
-6.42 ± 0.10	8.35 ± 0.29	$-5.7^{+1.1}_{-1.2}$	$0.0^{+0.5}_{-0.6}$	this work	OPAL'99 + DV_{V-A}
-6.48 ± 0.05	8.40 ± 0.18	$-3.6^{+0.7}_{-0.6}$	-1.0 ± 0.4	this work	ALEPH'14 + DV_{V-A}

$$R_\tau = N_C\, S_{EW}\, (1 + \delta_P + \delta_{NP})$$

Perturbative ($m_q=0$)

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

$K_0 = K_1 = 1$, $K_2 = 1.63982$, $K_3 = 6.37101$, $K_4 = 49.07570$

Baikov-Chetyrkin-Kühn ’08

$$\Rightarrow \quad \delta_P = \sum_{n=1} K_n\, A^{(n)}(\alpha_s) = a_\tau + 5.20\, a_\tau^2 + 26\, a_\tau^3 + 127\, a_\tau^4 + \cdots$$

Le Diberder- Pich ‘92

$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-2x+2x^3-x^4)\left(\frac{\alpha_s(-s)}{\pi}\right)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

$$R_\tau = N_C\, S_{EW}\, (1 + \delta_P + \delta_{NP})$$

Perturbative ($m_q=0$)

$$-s\,\frac{d}{ds}\,\Pi^{(0+1)}(s) = \frac{1}{4\pi^2}\sum_{n=0} K_n \left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

$K_0 = K_1 = 1$, $K_2 = 1.63982$, $K_3 = 6.37101$, $K_4 = 49.07570$ **Baikov-Chetyrkin-Kühn '08**

$$\delta_P = \sum_{n=1} K_n\, A^{(n)}(\alpha_s) = a_\tau + 5.20\, a_\tau^2 + 26\, a_\tau^3 + 127\, a_\tau^4 + \cdots$$

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$$A^{(n)}(\alpha_s) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-2x+2x^3-x^4)\left(\frac{\alpha_s(-s)}{\pi}\right)^n = a_\tau^n + \cdots \quad ; \quad a_\tau \equiv \alpha_s(m_\tau)/\pi$$

Power Corrections

$$\Pi^{(0+1)}_{\mathrm{OPE}}(s) \approx \frac{1}{4\pi^2}\sum_{n\geq 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-s)^n}$$

Braaten-Narison-Pich '92

$$C_4\langle O_4\rangle \approx \frac{2\pi}{3}\langle 0|\alpha_s G^{\mu\nu}G_{\mu\nu}|0\rangle$$

$$\delta_{NP} \approx \frac{-1}{2\pi i}\oint_{|x|=1} dx\,(1-3x^2+2x^3)\sum_{n\geq 2}\frac{C_{2n}\langle O_{2n}\rangle}{(-x m_\tau^2)^n} = -3\,\frac{C_6\langle O_6\rangle}{m_\tau^6} - 2\,\frac{C_8\langle O_8\rangle}{m_\tau^8}$$

Suppressed by m_τ^6 **[additional chiral suppression in $C_6\langle O_6\rangle^{V+A}$]**

α_s determination with ALEPH-like fit

Rodríguez-Sánchez, A.P.

$$\omega_{kl}(s) = \left(1-\frac{s}{m_\tau^2}\right)^{2+k}\left(\frac{s}{m_\tau^2}\right)^{l}\left(1+\frac{2s}{m_\tau^2}\right)$$

$(k,l)=(0,0) \quad \rightarrow \quad \alpha_s\,,\,\mathcal{O}_{6\,V/A}\,,\,\mathcal{O}_{8\,V/A}$

$(k,l)=(1,0) \quad \rightarrow \quad \alpha_s\,,\,\langle a_s GG\rangle\,,\,\mathcal{O}_{6\,V/A}\,,\,\mathcal{O}_{8\,V/A}\,,\,\mathcal{O}_{10\,V/A}$

$(k,l)=(1,1) \quad \rightarrow \quad \alpha_s\,,\,\langle a_s GG\rangle\,,\,\mathcal{O}_{6\,V/A}\,,\,\mathcal{O}_{8\,V/A}\,,\,\mathcal{O}_{10\,V/A}\,,\,\mathcal{O}_{12\,V/A}$

$(k,l)=(1,2) \quad \rightarrow \quad \alpha_s\,,\,\mathcal{O}_{6\,V/A}\,,\,\mathcal{O}_{8\,V/A}\,,\,\mathcal{O}_{10\,V/A}\,,\,\mathcal{O}_{12\,V/A}\,,\,\mathcal{O}_{14\,V/A}$

$(k,l)=(1,3) \quad \rightarrow \quad \alpha_s\,,\,\mathcal{O}_{8\,V/A}\,,\,\mathcal{O}_{10\,V/A}\,,\,\mathcal{O}_{12\,V/A}\,,\,\mathcal{O}_{14\,V/A}\,,\,\mathcal{O}_{16\,V/A}$

Channel	$\alpha_s(m_\tau^2)/\pi$	$\langle \frac{\alpha_s}{\pi} GG\rangle$ (10^{-3} GeV^4)	$\mathcal{O}_6$ (10^{-3} GeV^6)	$\mathcal{O}_8$ 10^{-3} GeV^8)
V (FOPT)	$0.328\,^{+0.013}_{-0.007}$	$8\,^{+7}_{-14}$	$-3.2\,^{+0.8}_{-0.5}$	$5.0\,^{+0.4}_{-0.7}$
V (CIPT)	$0.352\,^{+0.013}_{-0.011}$	$-8\,^{+7}_{-7}$	$-3.5\,^{+0.3}_{-0.3}$	$4.9\,^{+0.4}_{-0.5}$
A (FOPT)	$0.304\,^{+0.010}_{-0.007}$	$-15\,^{+5}_{-8}$	$4.4\,^{+0.5}_{-0.4}$	$-5.8\,^{+0.3}_{-0.4}$
A (CIPT)	$0.320\,^{+0.011}_{-0.010}$	$-25\,^{+5}_{-5}$	$4.3\,^{+0.2}_{-0.2}$	$-5.8\,^{+0.3}_{-0.3}$
V+A (FOPT)	$0.319\,^{+0.010}_{-0.006}$	$-3\,^{+6}_{-11}$	$1.3\,^{+1.4}_{-0.8}$	$-0.8\,^{+0.4}_{-0.7}$
V+A (CIPT)	$0.339\,^{+0.011}_{-0.009}$	$-16\,^{+5}_{-5}$	$0.9\,^{+0.3}_{-0.4}$	$-1.0\,^{+0.5}_{-0.7}$

Good agreement with Davier et al. (arXiv:1312.1501)

❶ Fit one more condensate to test stability/uncertainties

Channel	$\alpha_s(m_\tau^2)/\pi$	$\langle \frac{\alpha_s}{\pi} GG \rangle$ $(10^{-3}\ \mathrm{GeV}^4)$	$\mathcal{O}_6$ $(10^{-3}\ \mathrm{GeV}^6)$	$\mathcal{O}_8$ $10^{-3}\ \mathrm{GeV}^8)$	$\mathcal{O}_{10}$ $(10^{-3}\ \mathrm{GeV}^{10})$
V (FOPT)	$0.320\,^{+\,0.016}_{-\,0.014}$	$10\,^{+\,9}_{-\,17}$	$-4\,^{+\,3}_{-\,2}$	$6\,^{+\,2}_{-\,2}$	$-2\,^{+\,5}_{-\,5}$
V (CIPT)	$0.337\,^{+\,0.020}_{-\,0.019}$	$-1\,^{+\,10}_{-\,10}$	$-5\,^{+\,2}_{-\,2}$	$6\,^{+\,2}_{-\,2}$	$-4\,^{+\,4}_{-\,4}$
A (FOPT)	$0.347\,^{+\,0.022}_{-\,0.021}$	$-31\,^{+\,16}_{-\,33}$	$11\,^{+\,5}_{-\,4}$	$-12\,^{+\,4}_{-\,4}$	$15\,^{+\,9}_{-\,9}$
A (CIPT)	$0.373\,^{+\,0.029}_{-\,0.029}$	$-50\,^{+\,18}_{-\,16}$	$10\,^{+\,3}_{-\,3}$	$-11\,^{+\,3}_{-\,3}$	$14\,^{+\,7}_{-\,7}$
V+A (FOPT)	$0.333\,^{+\,0.013}_{-\,0.012}$	$-8\,^{+\,10}_{-\,24}$	$7\,^{+\,7}_{-\,4}$	$-5\,^{+\,4}_{-\,6}$	$12\,^{+\,12}_{-\,9}$
V+A (CIPT)	$0.355\,^{+\,0.016}_{-\,0.015}$	$-23\,^{+\,10}_{-\,8}$	$5\,^{+\,3}_{-\,3}$	$-5\,^{+\,3}_{-\,3}$	$10\,^{+\,8}_{-\,8}$

- **Good stability of α_s with respect to previous fit**
- **Larger variation in condensates values and increased errors**

❷ Take central values from first fit, adding differences as errors

α_s determination with ALEPH-like fit

Rodríguez-Sánchez, A.P.

$\omega_{kl}(x) = (1-x)^{2+k}\, x^l\, (1+2x)$, $x = s/m_\tau^2$, $(k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

Channel	$\alpha_s(m_\tau^2)/\pi$	$\langle \frac{\alpha_s}{\pi} GG \rangle$ $(10^{-3}\ \mathrm{GeV}^4)$	$\mathcal{O}_6$ $(10^{-3}\ \mathrm{GeV}^6)$	$\mathcal{O}_8$ $10^{-3}\ \mathrm{GeV}^8)$
V (FOPT)	$0.328\,^{+0.013}_{-0.007}$	$8\,^{+7}_{-14}$	$-3.2\,^{+0.8}_{-0.5}$	$5.0\,^{+0.4}_{-0.7}$
V (CIPT)	$0.352\,^{+0.013}_{-0.011}$	$-8\,^{+7}_{-7}$	$-3.5\,^{+0.3}_{-0.3}$	$4.9\,^{+0.4}_{-0.5}$
A (FOPT)	$0.304\,^{+0.010}_{-0.007}$	$-15\,^{+5}_{-8}$	$4.4\,^{+0.5}_{-0.4}$	$-5.8\,^{+0.3}_{-0.4}$
A (CIPT)	$0.320\,^{+0.011}_{-0.010}$	$-25\,^{+5}_{-5}$	$4.3\,^{+0.2}_{-0.2}$	$-5.8\,^{+0.3}_{-0.3}$
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V+A (CIPT)	$0.339\,^{+0.011}_{-0.009}$	$-16\,^{+5}_{-5}$	$0.9\,^{+0.3}_{-0.4}$	$-1.0\,^{+0.5}_{-0.7}$

- **High sensitivity to α_s. Bad sensitivity to power corrections**
- **Cancellation in $\mathcal{O}_{6,V+A}$ confirmed. $V+A$ more reliable**
- $K_5 = 275 \pm 400,\ \mu^2 = (0.5, 2)\, m_\tau^2$
- Best values taken from $V+A$. Errors increased with sensitivity to $\mathcal{O}_{10}$

$$\alpha_s(m_\tau^2)^{\mathrm{CIPT}} = 0.339\,^{+0.019}_{-0.017}$$

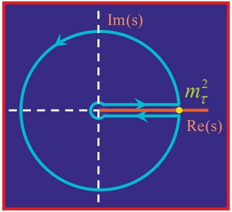
$$\alpha_s(m_\tau^2)^{\mathrm{FOPT}} = 0.319\,^{+0.017}_{-0.015}$$

→

$$\boxed{\alpha_s(m_\tau^2) = 0.329\,^{+0.020}_{-0.018}}$$

Good agreement with Davier et al.: $\alpha_s(m_\tau^2) = 0.332 \pm 0.012$ (arXiv:1312.1501)

$$A^{(n)}(a_\tau) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-2x+2x^3-x^4)\, a_\tau(-x m_\tau^2)^n \;=\; a_\tau^n + \cdots \qquad ; \qquad a_\tau \equiv \alpha_s(m_\tau)/\pi$$



$$A^{(1)}(a_\tau) = a_\tau - \frac{19}{24}\beta_1 a_\tau^2 + \left[\beta_1^2\left(\frac{265}{288}-\frac{\pi^2}{12}\right) - \frac{19}{24}\beta_2\right] a_\tau^3 + \cdots$$

$$a(-s) \simeq \frac{a_\tau}{1-\frac{\beta_1}{2}a_\tau \log\left(-s/m_\tau^2\right)} = \frac{a_\tau}{1-i\frac{\beta_1}{2}a_\tau\phi} = a_\tau \sum_n \left(i\frac{\beta_1}{2}a_\tau\phi\right)^n \qquad ; \qquad \phi\in[0,2\pi]$$

FOPT expansion only convergent if $\alpha_\tau < 0.14$ (0.11) [at 1 (3) loops]

Experimentally $\alpha_\tau \approx 0.11$ ➡ **FOPT should not be used** (divergent series)

FOPT suffers a large renormalization-scale dependence (Le Diberder- Pich , Menke)

The difference between FOPT and CIPT grows at higher orders

Renormalon Hypothesis: Asymptotics already reached

Modelling a better behaved FOPT

(Beneke – Jamin)

- **Large higher-order K_n corrections could cancel the g_n ones**

  Happens in the "large-β_0" approximation (UV renormalon chain)

- D = 4 corrections very suppressed in R_τ

  ➡ **n = 2 IR renormalons can do the job ($K_n \approx - g_n$)**

- No sign of renormalon behaviour in known coefficients

  ➡ **n = -1,2,3 renormalons + linear polynomial**

  5 unknown constants fitted to K_n ($2 \leq n \leq 5$). $K_5 = 283$ assumed

- **Borel summation:** large renormalon contributions. Smaller α

Nice model of higher orders. But too many different possibilities …

(Descotes-Genon – Malaescu)