Flyby-induced displacement: analytic solution

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Based on a joint work with P-M. Zhang, P. Horvathy

Report outline

- Nikiforov-Uvarov method.
- Scarf profile in displacement memory effect.
- Concluding remarks.

Nikiforov-Uvarov method

А.Ф. Никифоров, В.Б. Уваров, Специальные функции математической физики, Наука, Москва, 1978.

A. Nikiforov, V. Uvarov, Special Functions of Mathematical Physics: A Unified Introduction with Applications, Springer, Basel, 1988.

Second-order differential equations of generalized hypergeometric type:

$$u'' + \frac{\pi_1(z)}{\sigma(z)}\, u' + \frac{\sigma_1(z)}{\sigma^2(z)}\, u = 0, \tag{1}$$

can be solved using the Nikiforov-Uvarov technique. $\sigma(z)$ and $\sigma_1(z)$ are polynomials, at most of second degree, and $\pi_1(z)$ is a polynomial, at most of first degree.

Applications to quantum mechanics

Many problems in quantum mechanics can lead to equations of this type, and the Nikiforov-Uvarov method has been widely used in this context.

L. Ellis, I. Ellis, C. Koutschan, S. K. Suslov, On Potentials Integrated by the Nikiforov-Uvarov Method (2023).
arXiv:2303.02560.

C. Berkdemir, Application of the Nikiforov-Uvarov method in quantum mechanics, in: M. R. Pahlavani (Ed.), Theoretical Concepts of Quantum Mechanics, IntechOpen, Rijeka, 2012, Ch. 11, pp. 225–252.
doi:10.5772/33510

The first step

Solutions to equation (1) is invariant under a certain kind of “gauge"transformation

$$u(z) = e^{\varphi(z)} y(z),$$

if

$$\varphi' = \frac{\pi(z)}{\sigma(z)},$$

where $\pi(z)$ is some polynomial, at most of first degree.
The function $y(z)$ is also of generalized hypergeometric type:

$$y'' + \frac{\tau(z)}{\sigma(z)} y' + \frac{\sigma_2(z)}{\sigma^2(z)} y = 0,$$

where

$$\tau(z) = \pi_1(z) + 2\pi(z)$$

is a polynomial, at most of first degree, and

$$\sigma_2(z) = \sigma_1(z) + \pi^2(z) + \pi(z)\left[\pi_1(z) - \sigma'(z)\right] + \pi'(z)\sigma(z)$$

is a polynomial, at most of second degree.

The second step

Use the freedom in selecting the polynomial $\pi(z)$.

In particular, the equation (1) reduces to an equation of hypergeometric type

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, \tag{2}$$

if $\pi(z)$ is selected so that

$$\sigma_2(z) = \lambda\sigma(z),$$

where λ is a constant.

Finding the polynomial $\pi(z)$

$\pi(z)$ is the root of a quadratic equation and has the following form

$$\pi(z) = \frac{\sigma' - \pi_1}{2} \pm \sqrt{\left(\frac{\sigma' - \pi_1}{2}\right)^2 - \sigma_1 + k\,\sigma},$$

where $k = \lambda - \pi'$ is another constant.
Only if

$$\sigma_3(z) = \left(\frac{\sigma' - \pi_1}{2}\right)^2 - \sigma_1 + k\,\sigma$$

is the square of a first degree polynomial, will $\pi(\mathrm{z})$ be a polynomial. In this case, $\sigma_3(\mathrm{z})$ has a double root and its discriminant is zero:

$$\Delta(\sigma_3) = 0.$$

This condition determines the constant k and hence the constant λ.

Connection with algebraic geometry

There is an interesting connection with algebraic geometry: finding the polynomial $\pi(z)$ corresponds to the primary decomposition of the ideal. In general, there will be four possible solutions for the combination (π, k) (there will be four primary ideals), and only one will be physically acceptable.

I. Kikuchi, A. Kikuchi, Explanation of Nikiforov-Uvarov method in quantum mechanics – with a view toward algebraic geometry (2020). Preprint doi:10.31219/osf.io/4mzye.

The third step

Find polynomial solutions of the hypergeometric type equation. Such solutions exist only for certain values of λ.

The derivatives of the function $y(z)$, $v_n(z) = y^{(n)}(z)$, $n = 0, 1, 2, \ldots$, also satisfy hypergeometric type equation:

$$\sigma(z)v_n'' + \tau_n(z)v_n' + \mu_n v_n = 0, \tag{3}$$

where

$$\tau_n(z) = n\sigma' + \tau, \quad \mu_n = \lambda + n\tau' + \frac{1}{2}n(n-1)\sigma''.$$

If $y(z) = y_n(z)$ is a polynomial of degree n, then v_n is a constant and the equation (3) will be fulfilled only if $\mu_n = 0$. Therefore, we will have polynomial solutions only if the following “quantization condition" is satisfied

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''.$$

Rodrigues formula

The polynomial $y_n(z)$ is given by the Rodrigues formula:

$$y_n(z) = \frac{B_n}{\rho(z)} \left[\sigma^n(z)\rho(z)\right]^{(n)},$$

where B_n is some (normalization) constant, and the weight functions $\rho(z)$ satisfies Pearson equation

$$(\sigma\rho)' = \rho\,\tau.$$

The last step

Find the gauge function $\varphi(z)$ by solving the equation

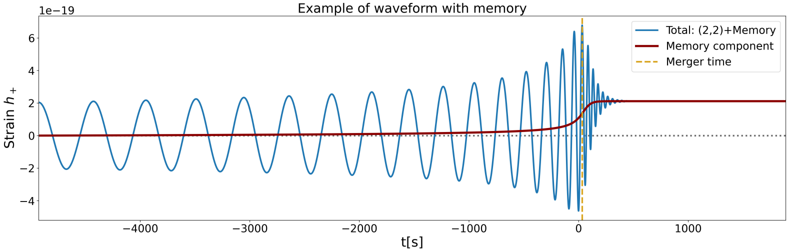
$$\varphi' = \frac{\pi(z)}{\sigma(z)}.$$

Scarf profile in displacement memory effect

P. M. Zhang, Z. K. Silagadze and P. A. Horvathy, “Flyby-induced displacement: analytic solution,"Phys. Lett. B 868 (2025) 139687

Gravitational displacement memory effect

- The so-called linear memory effect, was first predicted in 1974 by Zel'dovich and Polnarev.
- This linear effect manifests in the presence of, for example, astrophysically compact hyperbolic encounters.
- The non-linear memory effect, predicted in 1991 by Christodoulou, applies to all GW sources.



Does it exists?

G.W. Gibbons, The gravitational memory effect: what it is and why Stephen and I did not discover it. Talk at Stephen Hawking 75th Birthday Conference, Cambridge, UK, July 2-5, 2017.

“One may give many reasons why Stephen and I did not discover the memory effect

- We were too stupid and but this is obviously wrong.
- We were considering bar detectors and Zeldovich and Polanarev’s argument does not apply.
- No one had thought or built interferomometer detectors let alone had dreamt of satellite detectors.
- Even if we had thought about it we were only using linear theory and so any argument would have been unconvincing.
- There is no such effect.

IN FACT THE LAST TWO BULLET POINTS ARE TRUE! ...and numerical calculations bear this out."

Hyperbolic Scarf potential as flyby profile

In Brinkmann coordinates, the *pp*-gravitational wave metric is

$$ds^2 = d\mathrm{X}^2 + 2dUdV + A(U)\big(X_+^2 - X_-^2\big)\,dU^2\,,$$

and the geodesics are determined by the transverse equations

$$\frac{d^2X^\pm}{dU^2} \mp \frac{1}{2}A(U)X^\pm = 0\,.$$

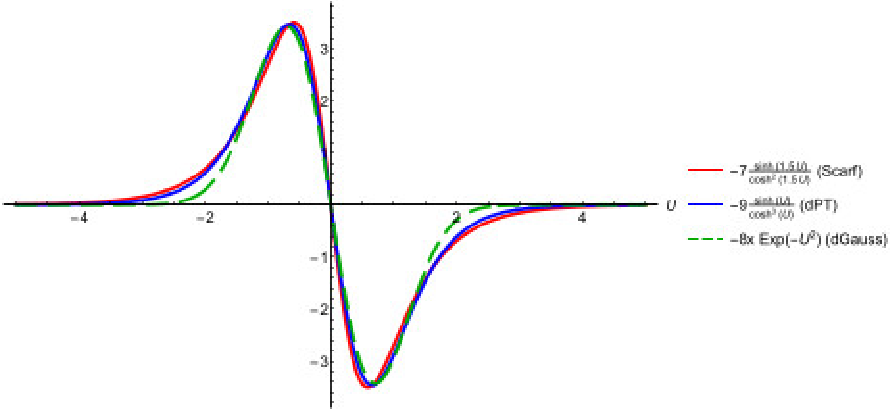
The flyby profile can be approximated by the hyperbolic Scarf potential

$$A^{Scarf}(U) = 2g\,\frac{d}{dU}\left(\frac{1}{\cosh U}\right) = -2g\,\frac{\sinh U}{\cosh^2 U}\,,$$

and the geodesic equation, putting $t = \sinh U$, takes the form

$$(1+t^2)\frac{d^2X^\pm}{dt^2} + t\,\frac{dX^\pm}{dt} \mp g\,\frac{t}{1+t^2}\,X^\pm = 0\,.$$

Hyperbolic Scarf potential versus derivative of Gaussian



Analysing geodesic equation

The geodesic equation, in the case of the the Scarf profile, is of the generalised hypergeometric type.

Thus NU method can be used to analyze it. In particular

$$\sigma = 1 + t^2,\ \pi_1 = t,\ \sigma_1 = -gt\,.$$

Then,

$$\sigma_3 = \frac{t^2}{4} + gt + k(1+t^2) = t^2\left(k + \frac{1}{4}\right) + gt + k\,.$$

Therefore, the condition $\det(\sigma_3) = g^2 - 4k\left(k + \frac{1}{4}\right) = 0$ gives

$$k = \frac{\pm\sqrt{1+16g^2} - 1}{8} \quad \text{and} \quad |g| = 2\sqrt{k\left(k + \frac{1}{4}\right)}\,.$$

Determination of π and τ polynomials

Thus we have

$$\sigma_3 = \frac{g^2}{4k}\left(t + \frac{2k}{g}\right)^2, \quad \pi = \frac{t}{2} \pm \sqrt{\sigma_3} = \frac{t}{2} \pm \frac{|g|}{2\sqrt{k}}\left|t + \frac{2k}{g}\right|.$$

Then

$$\tau = \pi_1 + 2\pi = 2t \pm \frac{|g|}{\sqrt{k}}\left|t + \frac{2k}{g}\right|.$$

To have real solutions, we assume $k > 0$, that is we choose the upper sign

$$k = \frac{\sqrt{1+16g^2} - 1}{8}.$$

“Magic"strengths

On the other hand, since

$$\tau' = 2 \pm \epsilon \, \frac{|g|}{\sqrt{k}} \ \text{ with } \ \epsilon = \mathrm{sign}\left(t + \frac{2k}{g}\right) ,$$

the quantization condition requires

$$\lambda = -n(n+1) \mp \epsilon \, \frac{n|g|}{\sqrt{k}} ,$$

and we get, after some algebra,

$$k = \lambda - \pi' = -n^2 - n - \frac{1}{2} \mp \epsilon \frac{|g|}{\sqrt{k}} \left(n + \frac{1}{2}\right) ,$$

and using $|g|/\sqrt{k} = 2\sqrt{k + 1/4}$, we get $\sqrt{k + \frac{1}{4}} = \mp\epsilon \left(n + \frac{1}{2}\right)$.
Therefore, we must choose signs so that $\mp\epsilon = 1$.
Then $k_n = n(n+1)$, and the “magic"strengths, labeled by the integer “quantum” number n, for which we have polynomial solutions $y_n(t)$, are

$$|g| = |g_n| = (2n+1)\sqrt{n(n+1)}.$$

Finding gauge function

It remains to find the gauge function $\varphi(t)$ and the corresponding hypergeometric type polynomials $y_n(t)$.
Since

$$\pi(t) = -nt - \mathrm{sign(g)}\sqrt{n(n+1)},$$
$$\tau(t) = -(2n-1)t - 2\,\mathrm{sign}(g)\sqrt{n(n+1)}\,,$$

the equation for the gauge function is

$$\frac{d\varphi}{dt} = \frac{\pi}{\sigma} = -\frac{nt + \mathrm{sign}(g)\sqrt{n(n+1)}}{1+t^2}\,,$$

Therefore, up to an irrelevant additive constant which translates to a multiplicative constant in the solution $X^{\pm}(t)$,

$$\varphi(t) = -\frac{n}{2}\ln(1+t^2) + \mathrm{sign}(g)\,\sqrt{n(n+1)}\,\mathrm{arctan}\,t\,.$$

$y_n(t)$ polynomials

The polynomial $y_n(t)$ is given by the Rodrigues formula, which requires to calculate the weight function $\rho(t)$, for which the equation is

$$\frac{(\sigma\rho)'}{\sigma\rho}=\frac{\tau}{\sigma}=\frac{-(2n-1)t-2\,\mathrm{sign}(g)\sqrt{n(n+1)}}{1+t^2},$$

with the solution

$$\rho(t)=(1+t^2)^{-\left(n+\frac{1}{2}\right)}e^{-2\,\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t}\,.$$

Then the geodesic trajectory is,

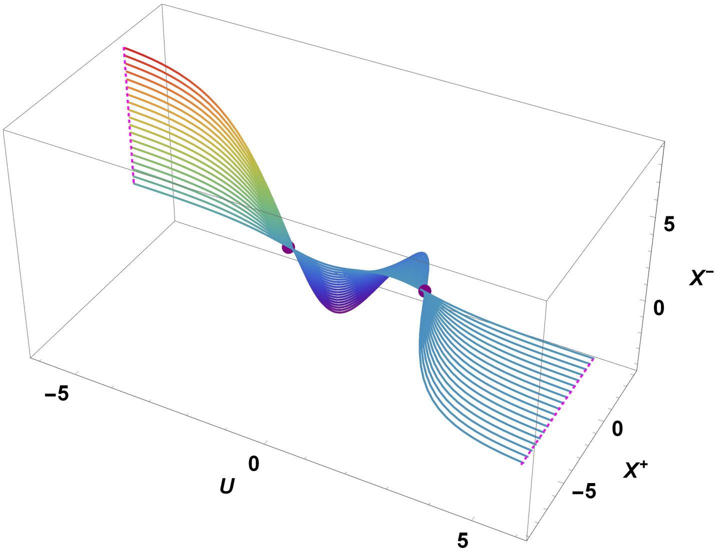
$$X_n^{\mathrm{sign}(g)}(t)=(1+t^2)^{-\frac{n}{2}}\,e^{-\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t}\,y_n(t)\,,$$

where

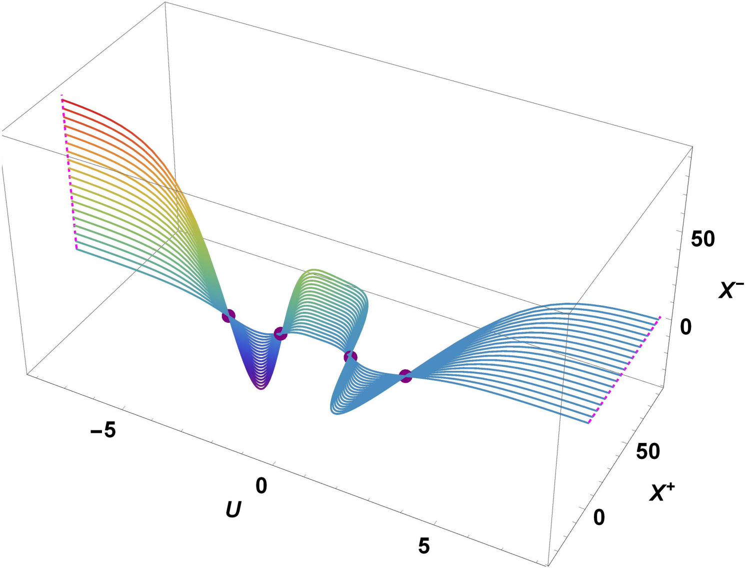
$$y_n(t)=\frac{B_n}{\rho(t)}\left[(1+t^2)^n\rho(t)\right]^{(n)}\,.$$

The normalization constants B_n is determined from the initial conditions.

The DM geodesics for the Scarf profile. For n = 1.



The DM geodesics for the Scarf profile. For $n = 2$.



Concluding remarks

- The Nikiforov-Uvarov method is a simple, yet elegant and powerful method for solving second-order differential equations of generalized hypergeometric type.
- Unfortunately, not all second-order differential equations are subject to this method.
- What I was talking about "remain mathematics without measurement and physics without experiment". (R. Hooykaas, Fact, Faith and Fiction in the Development of Science: The Gifford Lectures Given in the University of St Andrews 1976.)
- In general relativity, it is notoriously difficult to disentangle real physical effects from coordinate artifact.
- Research on the observational consequences of the displacement Memory Effect is a vast field with a rich literature.