ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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ИССЛЕДОВАНИЙ

BLTP

ОБЩИЙ ПОДХОД К УРАВНЕНИЯМ РЕНОРМГРУППЫ В ЛОКАЛЬНОЙ КВАНТОВОЙ ТЕОРИИ ПОЛЯ

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The foundation:

- Quantum field theory is formulated for all types of interactions independently on renormalizability

- R-operation equally works for NR theories and leads to local counter terms resulting in finite amplitudes

- Local Quantum Field Theory obeying the requirements of causality, unitarity and analyticity has a remarkable property: after applying the R-operation all UV divergent structures are local in coordinate space or are at most polynomial in momentum space (Bogoliubov-Parasiuk theorem).

- This is true for any local QFT irrespectively of its renormalizability or non-renormalizability

- All these statements lead to relations between the subsequent orders of PT resulting in the RG equations which aimed on the summation of infinite series of PT for the asymptotics of the Green functions, amplitudes, potential, etc

Based on:

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European Phys. J., 85 (2025) 1000, arXiv: 2504.03269 [hep-th]

Renormalization

Bogoliubov-Parasiuk-Hepp-Zimmermann R-operation

$$RG = \prod_{\gamma_i}(1 - M_i)G$$

$$RG = (1 - K)R'G$$

G - graph

γ - divergent subgraph

M_i - subtraction operator

K-operation extracts the singular part

Incomplete R-operation   A.N.Vasiliev, Green Book

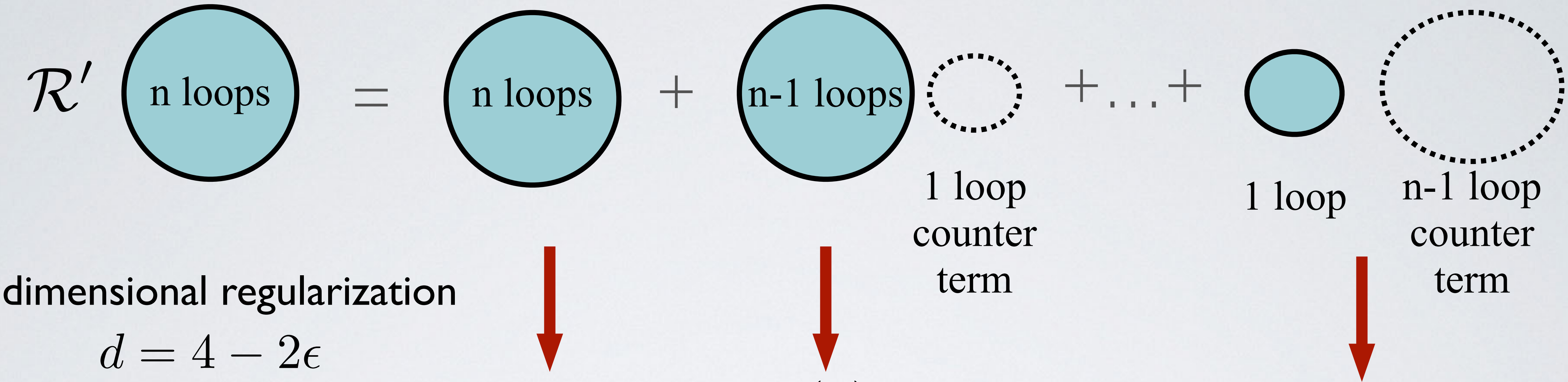
$$R' = 1 - \sum_{\gamma} KR'\gamma + \sum_{\gamma,\gamma'} KR'\gamma\, KR'\gamma' - \ldots;$$

R-operation is equivalent to the introduction of the counterterms into the Lagrangian

$$\mathcal{L} \Rightarrow \mathcal{L} + \Delta\mathcal{L}$$

Bogolyubov-Parasiuk Theorem: In any local quantum field theory after subtracting the UV divergences in subgraphs the resulting counterterms are always local in coordinate space or at most are polynomials of external momenta in momentum space in each order of perturbation theory

BPHZ R-operation



- dimensional regularization

$$d = 4 - 2\epsilon$$

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$$+ \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + \ldots + \frac{B_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-1}}$$

$$+ \frac{C_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-2}} + \frac{C_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-2}} + \ldots + \frac{C_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-2}}$$

$$+ \text{lower pole terms,}$$

$A_k^{(n)}$ $B_k^{(n)}$ $C_k^{(n)}$ $(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

BPHZ R-operation

<u>Bogoliubov-Parasiuk Theorem:</u> <u>(Locality)</u>

$R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

- Due to <u>locality</u> all higher order divergences are related to the lower ones

$$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n},$$
One loop

$$B_n^{(n)} = (-1)^n\left(\frac{2}{n}B_2^{(n)} + \frac{n-2}{n}B_1^{(n)}\right),$$
One and two loops

$$C_n^{(n)} = (-1)^{n+1}\left(\frac{3}{n}C_3^{(n)} + \frac{2(n-3)}{n}C_2^{(n)} + \frac{(n-2)(n-3)}{2n}C_1^{(n)}\right).$$
One, two and three loops

$$\mathcal{K}\mathcal{R}'G_n = \sum_{k=1}^{n}\left(\frac{A_k^{(n)}}{\epsilon^n} + \frac{B_k^{(n)}}{\epsilon^{n-1}} + \frac{C_k^{(n)}}{\epsilon^{n-2}} + \cdots\right) \equiv \frac{A_n^{(n)'}}{\epsilon^n} + \frac{B_n^{(n)'}}{\epsilon^{n-1}} + \frac{C_n^{(n)'}}{\epsilon^{n-2}} + \cdots .$$

$$A_n^{(n)'} = (-1)^{n+1}A_n^{(n)} = \frac{A_1^{(n)}}{n},$$
One loop

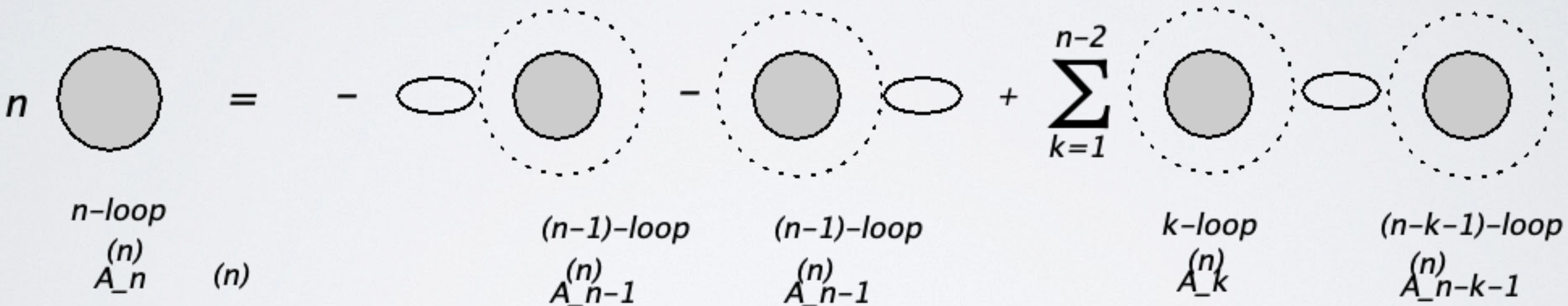
$$B_n^{(n)'} = \left(\frac{2}{n(n-1)}B_2^{(n)} + \frac{2}{n}B_1^{(n)}\right),$$
One and two loops

$$C_n^{(n)'} = \left(\frac{2}{(n-1)(n-2)}\frac{3}{n}C_3^{(n)} + \frac{2}{n-1}\frac{3}{n}C_2^{(n)} + \frac{3}{n}C_1^{(n)}\right).$$
One, two and three loops

The Recurrence Relations

- These properties allow one to write down the recurrence relations connecting the subsequent orders of the counterterms and to evaluate them algebraically without calculating the diagrams. This can be done in renormalizable and non-renormalizable theories. The difference is a more complicated structure of these relations in NR case.

Leading divergences:



- These recurrence relations can be promoted to the RG equations for the scattering amplitudes, effective potential, etc which sum up the leading divergences (logarithms) and to find out the high energy/field behaviour

Examples:

- Maximally supersymmetric gauge theory in D=6,8,10 dimensions SYM_D
- Scalar field theory in D=4,6,8,10 dimensions ϕ^4_D
- Four-fermion interaction in D=4
- Supersymmetric Wess-Zumino model with quartic superpotential in D=4 Φ^4_4

ϕ_6^4

UV divergences within dim reg

Loop Expansion (non-renormalizable case)

$$\Delta\mathcal{L}_1 \sim \lambda^2(s+t+u)\Phi^4(\frac{1}{\epsilon}+c_{11})$$

Momentum space

$$\Delta\mathcal{L}_1 \sim \lambda^2\partial^2\Phi^2\Phi^2(\frac{1}{\epsilon}+c_{11}),$$

Coordinate space

Increasing power of momentum

$$\Delta\mathcal{L} = \underset{(\frac{1}{\epsilon}+c_{11})}{\lambda^2\partial^2\Phi^2\Phi^2} + \lambda^3[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{12})}{\partial^4\Phi^2\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{13})}{\partial^2\Phi^2\partial^2\Phi^2)}] + \lambda^4[...] + \lambda^5[...]$$

$$\searrow \qquad \searrow \qquad \searrow$$

$$\underset{(\frac{1}{\epsilon}+c_{21})}{\lambda^3\Phi^6} + \lambda^4[\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{22})}{\partial^2\Phi^4\Phi^2} + \underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{23})}{\partial^2\Phi^2\Phi^4}]$$

$$\searrow \qquad \searrow$$

$$\underset{(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{32}),}{\lambda^5\Phi^8}$$

Increasing power of fields

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SYM_D

D=6 N=2

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_3 = -s/3,\ T_3 = -t/3$

$$nS_n(s,t) = -2s\int_0^1 dx \int_0^x dy\ (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$n \geq 4$

$t' = t(x-y) - sy$

D=8 N=1

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

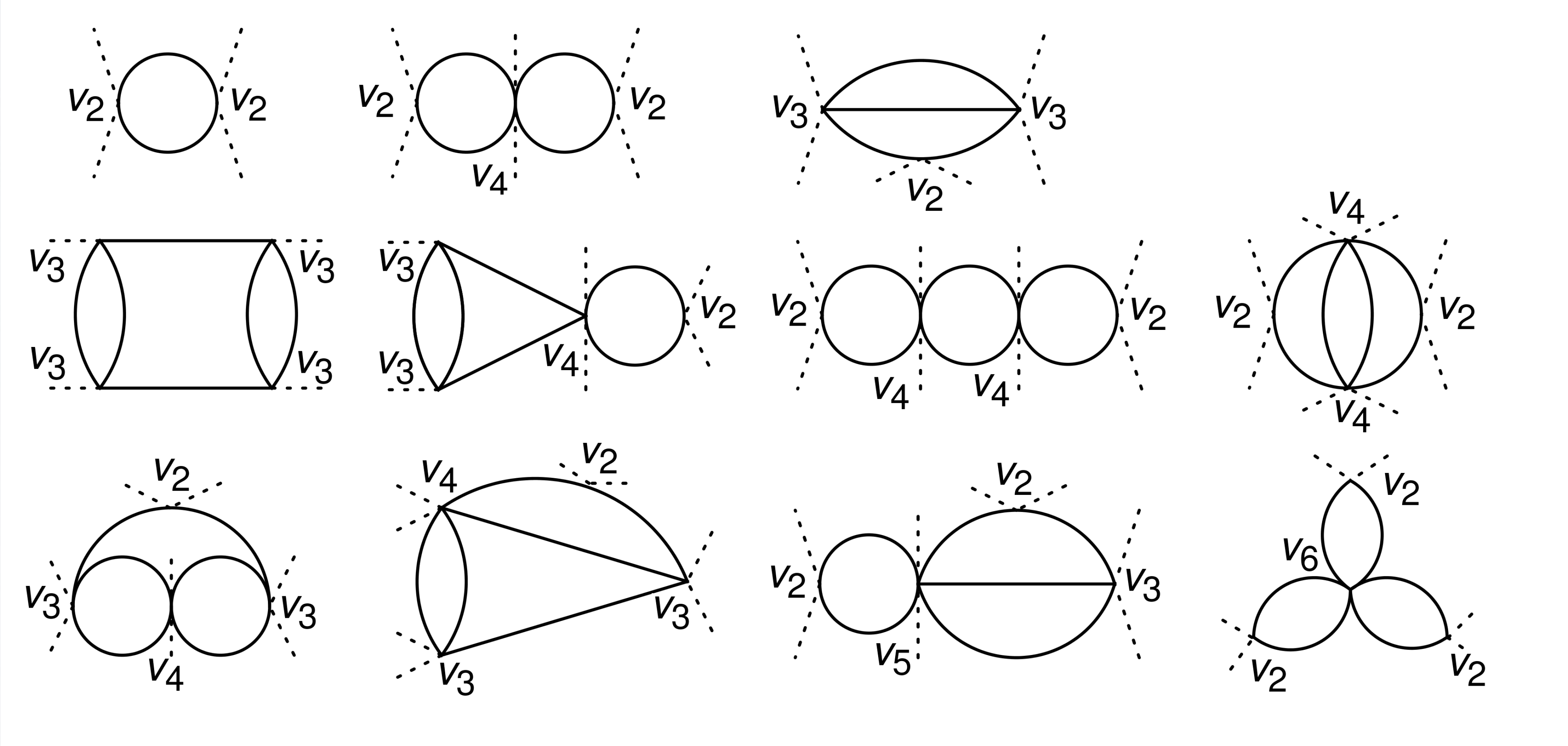
Exact all-loop recurrence relation $S_1 = \frac{1}{12},\ T_1 = \frac{1}{12}$

$$nS_n(s,t) = -2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu}$$
$$+ \quad s^4\int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2}\sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!}\ \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$
$$\times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

Effective Potential in Scalar Theory

V_{eff} Is the sum of all vacuum 1PI diagrams

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - g V_0(\phi) \qquad V_{eff} = g \sum_{n=0}^{\infty} (-g)^n V_n .$$

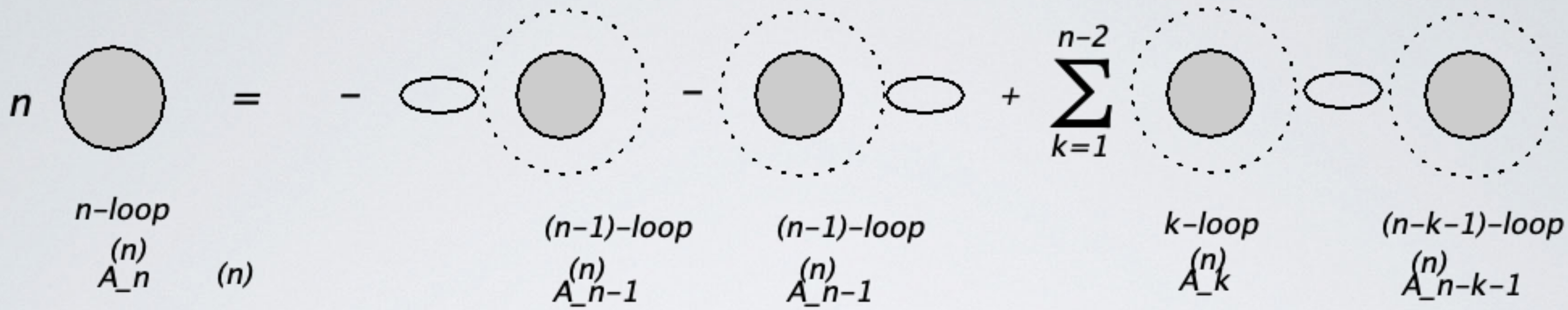


$$v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}$$

$$v_n \equiv d^n V_0 / d\phi^n$$

$$n \Delta V_n = \frac{1}{2} v_2 D_2 \Delta V_{n-1} + \frac{1}{4} \sum_{k=1}^{n-2} D_2 \Delta V_k D_2 \Delta V_{n-1-k}, \quad n \geq 2$$

From Recurrence Relation to the RG Equation



- This is the general recurrence relation that reflects the locality of counterterms in any theory
- In <u>renormalizable</u> theories A_n is a constant and this relation is reduced to the algebraic one
- In <u>non-renormalizable</u> theories for the scattering amplitudes A_n depends on kinematics and one has to integrate through the one-loop diagrams
- For the effective potential A_n depends on the fields and one has to differentiate

Taking the sum $\sum_n A_n(-z)^n = A(z)$ one can transform the recurrence relation into RG equation

$$\frac{dA(z)}{dz} = -2 \oint\!\!\!\!\!\!\!D\, A(z) + \oint\!\!\!\!\!\!\!D\, A(z) \otimes A(z) \qquad \frac{d}{dz} = \frac{d}{d\log\mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

RG Equation

SYM_D

D=6 N=2

$$\Sigma(s,t,z) = z^{-2}\sum_{n=3}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s\int_0^1 dx\int_0^x dy\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=xt+yu}$$

Linear equation

D=8 N=1

$$\Sigma(s,t,z) = \sum_{n=1}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2\int_0^1 dx\int_0^x dy\ y(1-x)\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=tx+yu}$$
$$-s^4\int_0^1 dx\ x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}(\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

Non-linear equation

RG pole equation for arbitrary potential

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty} (-z)^n \Delta V_n(\phi) \qquad z = \frac{g}{\epsilon}$$

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RG pole equation

$$\frac{d\Sigma}{dz} = -\frac{1}{4}(D_2\Sigma)^2 \qquad \Sigma(0,\phi) = V_0(\phi)$$

This a non-linear partial differential equation!

Effective potential

$$V_{eff}(g,\phi) = g\Sigma(z,\phi)|_{z\to -\frac{g}{16\pi^2}\log g v_2/\mu^2}. \qquad v_2(\phi) \equiv \frac{d^2V_0(\phi)}{d\phi^2}$$

V-A four fermion Interaction

Lagrangian $$\mathcal{L} = i\bar{\Psi}\hat{\partial}\Psi - \frac{G}{\sqrt{2}}(\bar{\Psi}\hat{\mathcal{O}}\Psi)(\bar{\Psi}\hat{\mathcal{O}}\Psi) \qquad \hat{\mathcal{O}} = \gamma^{\mu}(1-\gamma^5)/2$$

Tree level amplitude $$A_4^{(0)} = \langle 13\rangle[42]$$

One-loop amplitude $$A_4^{(1)} = S_1 + T_1 + U_1 = \left(-\frac{16}{3}\frac{s}{\epsilon} - \frac{16}{3}\frac{t}{\epsilon}\right)\langle 13\rangle[42]$$

$$nS_n(s,t,u) = -4s\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \times$$
$$\times \frac{d^pA_k(s,-s-u',u')}{du'^p}\frac{d^pA_{n-1-k}(s,-s-u',u')}{du'^p}\Big|_{u'\to -sx}$$

$$nT_n(s,t,u) = -4t\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[t(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \times$$
$$\times \frac{d^pA_k(-t-u',t,u')}{du'^p}\frac{d^pA_{n-1-k}(-t-u',t,u')}{du'^p}\Big|_{u'\to -tx}$$

$$A_n(s,t,u) = S_n(s,t,u) + T_n(s,t,u) + U_n(s,t,u)$$

RG Equation for the Scattering Amplitude

$$A(s,t,u)=\sum_{n=0}^{\infty}A_n(s,t,u)(-z)^n$$

$$\begin{aligned}
-\frac{dA(s,t,u)}{dz} &= \\
&= -4s\int_0^1 dx\sum_{p=0}^{\infty}\frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}}\left(\frac{d^pA(s,-s-u',u')}{du'^p}\right)^2\\
&- 4t\int_0^1 dx\sum_{p=0}^{\infty}\frac{[t(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}}\left(\frac{d^pA(-t-u',t,u')}{du'^p}\right)^2\\
&+4u\int_0^1 dx\sum_{p=0}^{\infty}\frac{[u(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}}\left(\frac{d^pA(s',-s'-u,u)}{ds'^p}\right)^2\\
&+4u\int_0^1 dx\sum_{p=0}^{\infty}\frac{[u(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}}\left(\frac{d^pA(-t'-u,t',u)}{dt'^p}\right)^2
\end{aligned}$$

$$z=G/\epsilon\to -G\log Q^2/\mu^2$$

- With the help of this equation one can find the asymptotic behaviour of the amplitude in the high energy regime $s\sim t\sim u\sim E^2\to\infty$

Numerical Solution of RG Equation for the Scattering Amplitude

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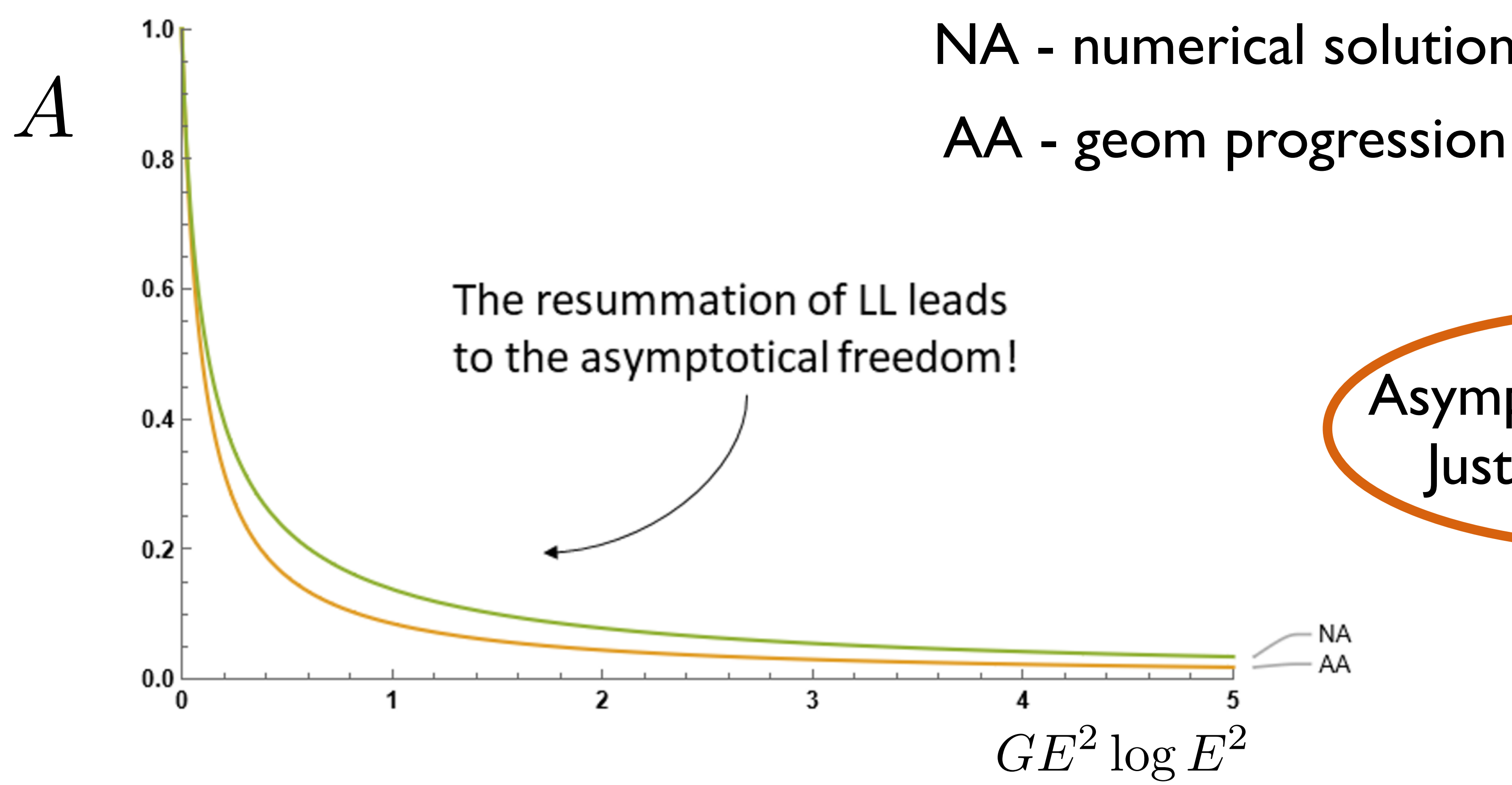
$$s = 4E^2, \; t = u = -2E^2, \; E \to \infty \qquad \frac{Amplitude}{A_{Tree}} = A(E^2, G \log E^2/\mu^2)$$



Asymptotic freedom !!
Just like in the SM

$$A_{Tree} \sim GE^2 \qquad A \sim \frac{1}{GE^2 \log E^2} \qquad Amp = A_{Tree} A \sim \frac{1}{\log E^2}$$

α -attractor Inflaton Potential

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Inflaton action with hyperbolic geometry

Kallosh, Linde 13

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R(g) + \frac{1}{2} \frac{\partial_\mu \phi \partial^\mu \phi}{1 - \frac{\phi^2}{6\alpha}} - V(\phi) \right]$$

Transition to the standard kinetic term $\partial\phi/\sqrt{1 - \frac{\phi^2}{1-6\alpha}} = \partial\varphi$ $\qquad \phi = \sqrt{6\alpha} \tanh\left(\frac{\varphi}{\sqrt{6\alpha}}\right)$

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R(g) + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V\left(\sqrt{6\alpha} \tanh\left(\frac{\varphi}{\sqrt{6\alpha}}\right)\right)\right].$$

T_2 - model $\quad V(\phi) = g\phi^2 \quad \rightarrow \quad V_T(\varphi) = \tanh^2\left(\frac{\varphi}{\sqrt{6\alpha} M_{Pl}}\right)$

$$\frac{d\Sigma}{dz} = -\frac{1}{4}(D_2\Sigma)^2$$

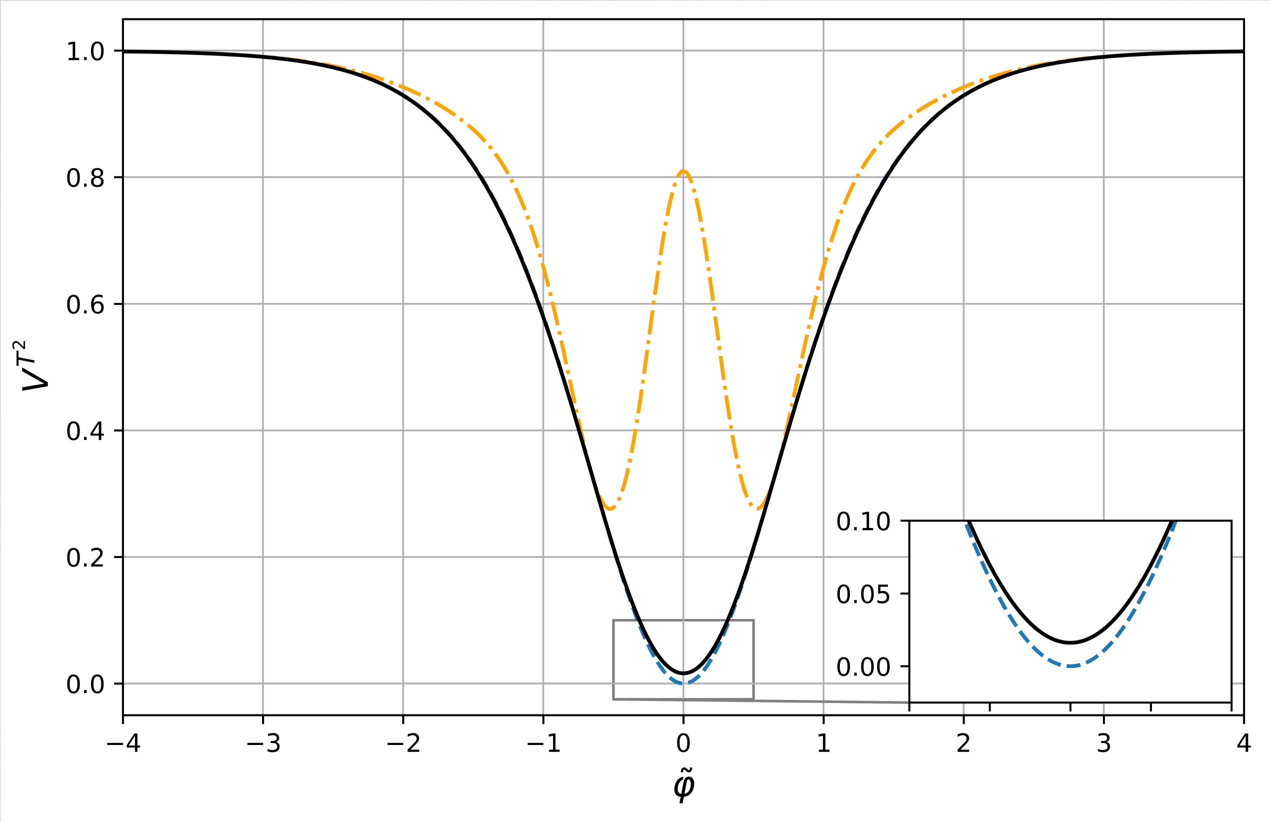
Dimensionless variables $\quad x = z/M_{Pl}^4 \quad y = \tanh^n(\varphi/\sqrt{6\alpha}M_{Pl})$

$$\Sigma(z/M_{Pl}^4, \tanh^n(\varphi/\sqrt{6\alpha}M_{Pl})) \equiv S(x,y)$$

$$S_x = -\frac{(y-1)^2\left((3y-1)S_y + 2(y-1)yS_{yy}\right)^2}{36\alpha^2}$$

$$S(0,y) = y, \;\; S(x,1) = 1, \;\; S_y(x,1) = 0.$$

Lift of the Potential at the Minima - Origin of the Cosmological Constant



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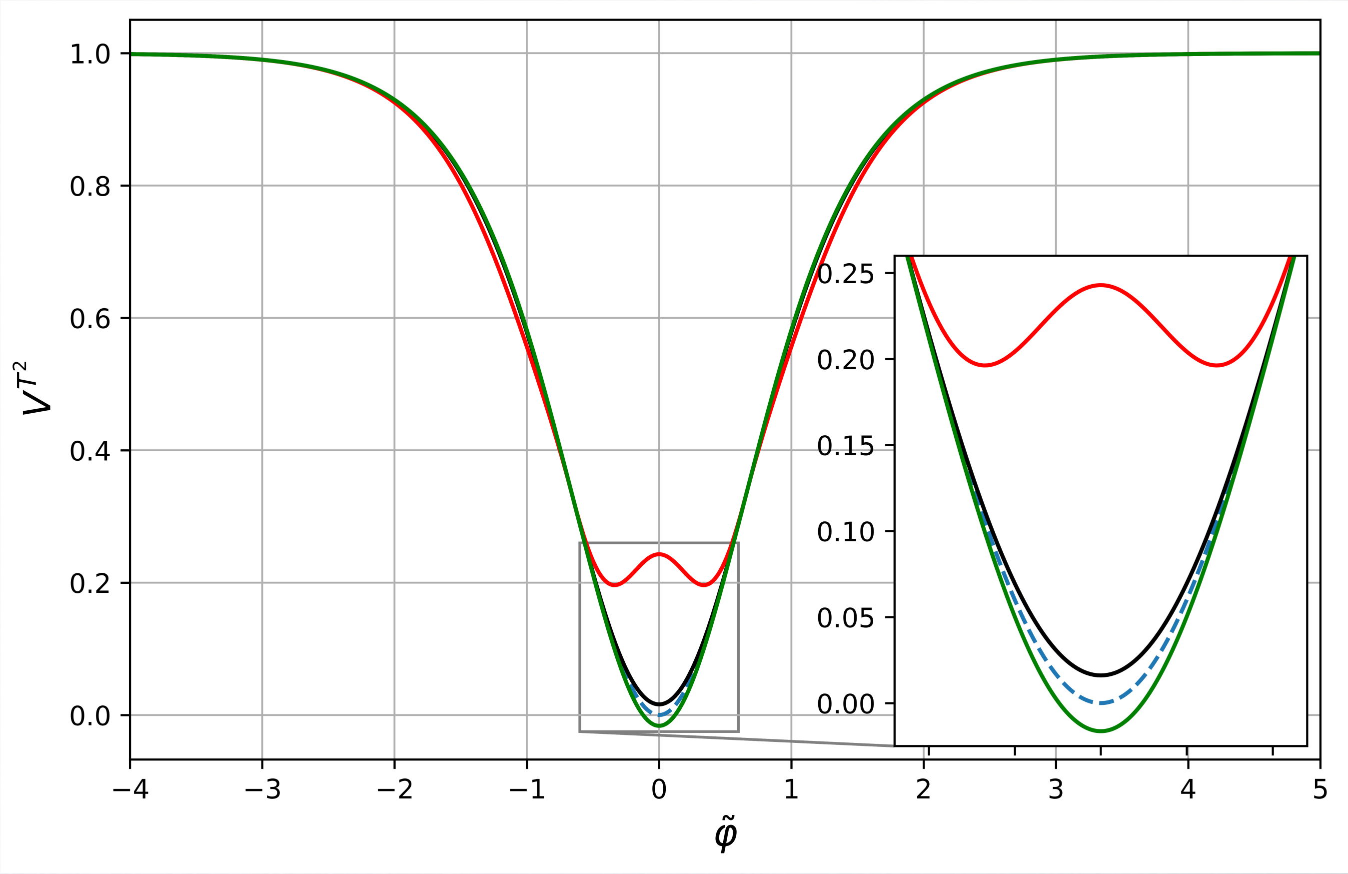
Comparison of the classical T2-model potential (blue dashed line), the one-loop correction (orange dashed line), and the RG summed potential (black solid line) for $g \sim 1, \mu < M_{PL}$

T2-model potential: variation of μ . The classical potential (blue dashed line), the RG summed potential (solid lines) for

$\mu < M_{Pl}$ $\mu \ll M_{Pl}$ $\mu > M_{Pl}$

black line, red line, green line

$$g = 2, \ \alpha = 1$$



Resume

- УФ-расходимости в неперенормируемых теориях являются локальными и могут быть устранены с помощью локальных контрчленов, как в перенормируем случае
- Основываясь на локальности контрчленов, обусловленной теоремой Боголюбова-Парасюка, можно построить рекуррентные соотношения для лидирующих, подлидирующих и т.д. расходимостей во всех петлях, начиная с одно-, двух-, трех- и т.д. петлевых диаграмм
- Рекуррентные соотношения могут быть преобразованы в обобщенные уравнения РГ точно так же, как в перенормируемых теориях
- Уравнения РГ позволяют суммировать лидирующие (подлидирующие) расходимости во всех петлях и находить поведение при высокой энергии
- Уравнения РГ для подлидирующих полюсов и т.д. всегда являются линейными дифференциальными уравнениями возрастающего порядка
- Уравнения РГ для логарифмов всегда являются дифференциальными уравнениями первого порядка с известными значениями в правой части