Discontinuities of multiparticle amplitudes and Steinmann relations

V.S Fadin

Budker Institute of Nuclear Physics
Novosibirsk

Nuclear Physics Section of the Department of Physical Sciences of RAS
March 10-13, 2026, Novosibirsk, Russia

Contents

Although multiparticle amplitudes play an increasingly important role in high-energy physics, their analytical properties are relatively poorly known. At the same time, they are actively used in theory both for direct description of processes with a large multiplicity and in unitarity relations used for calculating amplitudes with a smaller number of particles.

Here I will talk about properties of discontinuities of multiparticle amplitudes in energy invariants of overlapping channels. Recall that two channels are called overlapping if they contain common particles, but neither is completely contained within the other.
More precisely, I will talk about one widely used property of these discontinuities, the absence of simultaneous discontinuities in overlapping channels.

This property underlies the currently accepted multi-Regge form of the multiple production amplitudes used in the

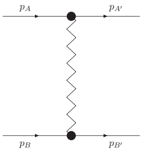
derivation of the Balitskii-Fadin-Kuraev-Lipatov (BFKL) equation.
V,S.F. E. A. Kuraev, L. N. Lipatov, Phys. Lett. B, Vol. 60, p. 50-52
For the derivation, analytic properties of these amplitudes are not important in the leading logarithmic approximation (LLA) and in the next-to the leading logarithmic approximation (NLLA), but become important in higher approximations.
It is the reason of my interest to this subject.

Remind that in the Regge-Gribov theory the asymptotics of the scattering amplitude $A(s,t)$ for $s \to \infty$ and a fixed t is determined by the position of the poles (called Reggeons) in the j-plane of the partial wave $A_l(t)$ analytically continued to complex j.
The analytic properties of the scattering amplitudes allow continuation from either even or odd l, so reggeons have an additional quantum number compared to particles

the signature. The contribution of a reggeon with trajectory $\alpha \equiv \alpha(t)$ and signature σ to the amplitude of the process $AB \to A'B'$ is given by the expression

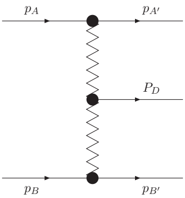
$$\mathcal{A}_{AB}^{A'B'} = \Gamma_{AA'}(t) s^{\alpha} \xi_{\alpha} \Gamma_{BB'}(t),$$

where $\Gamma_{AA'}(t)$ and $\Gamma_{AA'}(t)$ are the vertices of the reggeon-particle interaction, $s = (p_A + p_B)^2, \;\; t = (p_A - p'_A)^2$, $\xi_{\alpha} = \frac{e^{-i\pi\alpha} + \sigma}{\sin \pi\alpha}$ – signature factor and is presented by the picture



To create a Regge theory of multiparticle processes, knowledge of the analytical properties of many-particle amplitudes was required. In the absence of any reliably established properties of these amplitudes, various models were used.

It was recognized that the reggeon-reggeon-particle vertex $V^{D}_{R_1 R_2}(q_1, q_2)$ in the direct generalization of the regge pole contribution to the elastic amplitude for the case of the process $AB \to A' DB'$



$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1) s_1^{\alpha_1} \xi_{\alpha_1} V^{D}_{R_1 R_2}(q_1, q_2) s_2^{\alpha_2} \xi_{\alpha_2} \Gamma_{BB'}(t_2),$$

in the Multi-Regge kinematics

$$s \gg s_i \gg |t_i|, \quad i = 1, 2, \quad s = (p_A + p_B)^2, \quad s_1 = (p'_A + p_D)^2,$$

$$s_2 = (p'_B + p_D)^2, \quad t_1 = (p_A - p'_A)^2, t_2 = (p_B - p'_B)^2$$

has a complicated analytical structure.
Finally, Regge theory for multiparticle amplitudes was built on the system of postulates.
R. C. Brower, Carleton E. DeTar, and J. H. Weis, Regge Theory for Multiparticle Amplitudes, Phys. Rept., 14:257, 1974.
One of the most important postulated properties of many-particle amplitudes is the absence of simultaneous discontinuities in the squares of the invariant masses of overlapping channels.
To justify this hypothesis, reference is made usually to the Steinmann relations
O. Steinmann, Uber den Zusammenhang zwischen den Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 267-298; Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 347-362.

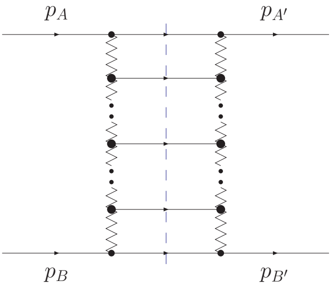
This property allows one to write a multi-Regge representation for many-particle amplitudes in a form that explicitly shows all their analytical properties. Contribution of Reggeons with trajectories $\alpha_i(t)$ and signatures τ_i to the amplitude of the process $AB \to A'CB'$
is presented in the form

$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1)\Big[s^{\alpha_2}\xi_{\alpha_2}s_1^{\alpha_1-\alpha_2}\xi_{\alpha_1\alpha_2}V_R(t_1,t_2,\kappa)$$

$$+s^{\alpha_1}\xi_{\alpha_1}s_2^{\alpha_2-\alpha_1}\xi_{\alpha_2\alpha_1}V_L(t_1,t_2,\kappa)\Big]\Gamma_{BB'}(t_2),$$

where $\kappa = \frac{s_1 s_2}{s}, \quad \xi_{\alpha_1\alpha_2} = \frac{e^{-i\pi(\alpha_1-\alpha_2)}+\tau_1\tau_2}{\sin(\pi(\alpha_1-\alpha_2))}$.
The advantage of this representation is that for $t_i < 0$ the functions V_L and V_R are real, which allows us to uniquely separate the amplitude into real and imaginary parts. Similar representations exist for the production of a larger number of particles (with the number of vertices increasing with the number of particles).

The BFKL approach is founded on the gluon Reggeization.
In the dispersive method, used for the derivation of the BFKL equation, the unitarity relations are used for the calculation of imaginary parts of elastic amplitudes. Regge form of multiparticle amplitudes is used in unitarity relations.
In the unitarity relations, multiple production amplitudes in the multi-Regge kinematics (MRK) must be taken into account.
MRK is the kinematics where all particles have limited transverse momenta (with respect to momenta of colliding particle) and are combined into jets with limited invariant mass of each jet and large (increasing with s) invariant mass of any pair of jets.
The muli-Regge form was used for these amplitudes.



The s-channel discontinuity.

The fallacy of the hypothesis of the absence of simultaneous discontinuities in overlapping channels, and hence of the multi-Regge form of multiparticle amplitudes based on this hypothesis, may cast doubt on the derivation of the BFKL equation.

However, these doubts are unfounded both in the LLA and in the NLLA since in these approximations only the real part of the amplitudes included in the unitarity relations was used in deriving the BFKL equation.
It is quite clear in the LLA, where imaginary parts of the multiparticle amplitudes are neglected.
This is also true in the NLLA.
The reason is that in this approximation one of two amplitudes in the unitarity relations can lose $\ln s$, while the second one must be taken in the LLA. The LLA amplitudes are real, so that only real parts of the NLLA amplitudes are important in the unitarity relations.

As it was said already, the Steinmann relations are used to justify the statement of absence of simultaneous discontinuities of multiparticle amplitudes in energy invariants of overlapping channels.

But originally the Steinmann relations have few common with this statement.

The Steinmann's papers were devoted to investigation of connection between two approaches to axiomatic quantum field theory: Wightman's one and the LSZ (Lehmann-Symanzik-Zimmerman) approach. In the first one, the consequences of the basic postulates of the theory for the system of the vacuum averages

$$W(x_0, ...x_{n-1}) =< A(x_0)...A(x_n)) >_0$$

of the products of field operators at arbitrary points in space-time were studied and it was shown that the system of all W uniquely defines the theory.

Unfortunately, the concept of S-matrix cannot be incorporated into this formalism. The approach of Lehman, Simanczyk, and Zimmerman is being made to construct the theory as a theory of the S-matrix. In this approach, the vacuum averages

$$r(x; x_1, x_2, ...x_n) =< R(x; x_1, x_2, ...x_n) >_0$$

from retarded products of field operators were studied.

$$n \geq 1 : R(x; x_1, x_2, ...x_n) = (-i)^n \sum_{\mathcal{P}(x_1, x_2, ...x_n)} \theta(x - x_1)\theta(x_1 - x_2)....$$

$$\theta(x_{n-1} - x_n)[...[A(x), A(x_1)]...A(x_n)].$$

The connection with the S-matrix is established using the so-called asymptotic conditions, a statement about the behavior of the field in the limit of $t \to \infty$.

The problem which was considered in Steinmann's papers is: under what conditions is the connection between $\{r_n\}$ and $\{W_n\}$ solvable with respect to $\{W_n\}$? Are the properties of $\{r_n\}$ without following from the asymptotic conditions are sufficient? If not, what properties are needed?

The Steinmann relations were obtained for the retarded commutators $\{r_n\}$, and not for matrix elements of the S matrix. Therefore, they can not be considered as justification of the the absence of simultaneous discontinuities in overlapping channels.

Actually, if such discontinuities are infrared singular, they can not be prohibited at all in axiomatic quantum field theories It was indicated by Steinmann himself. In particular, he wrote in O. Steinmann, Acta Phys. Austriaca Suppl. **11** (1973), 167-198

" In the known proofs of asymptotic conditions it is assumed that the particles under consideration belong to isolated one-particle hyperboloids in the energy-momentum spectrum. "

The factorisation of infrared singularities is well known in quantum electrodynamics (QED).
The amplitudes of processes with an arbitrary number of particles with momenta p_i (all momenta are considered incoming) are represented as

$$A(\{p_i\}) = \exp\left\{ -\sum_{i<j} Q_i Q_j V\left(p_i, p_j\right) \right\} A_{ns}(\{p_i\}),$$

$$V\left(p_i, p_j\right) = -\frac{e^2}{2} \int \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2-\lambda^2+i0} \left(\frac{2p_i-k}{k^2-2(kp_i)+i0} + \frac{2p_j+k}{k^2+2(kp_j)+i0} \right)^2,$$

where $Q_i = 1$ for an electron (positron) in the initial (final) state and $Q_i = -1$ for an electron (positron) in the final (initial) state, λ –introduced to regularize the infrared divergence of the "photon mass", and the amplitude $A_{ns}(\{p_i\})$ is finite at $\lambda \to 0$.

The expression for $V(p_i, p_j)$ integrated over d^4k is well known. Its infrared singular part is quite simple, especially in the high-energy region of interest to us:

$$V_{sing}(p_i, p_j) \simeq \frac{\alpha}{2\pi}\left(\ln\left(\frac{-s_{ij}}{m^2}\right) - 1\right)\ln\left(\frac{m^2}{\lambda^2}\right),$$

where $s_{ij} = (p_i + p_j)^2$. Since the exponent in the infrared singular factor contains the sum $\sum_{i<j}\ln(-s_{ij})$ over all channels, then when expanding the exponent we obtain products of powers of $\ln(-s_{ij})$ over all channels, including overlapping ones, i.e. terms that violate the prohibition on the existence of simultaneous discontinuities in overlapping channels.
In quantum chromodynamics (QCD), factorization is complicated by the non-Abelian nature of the theory, which leads to both additional singularities and to a matrix structure of the emission vertices. But in principle the situation is the same.

The existence of simultaneous discontinuities in the energy invariants of overlapping channels was verified
V.S. F., JETP Lett. **122** (2025) no.4, 203-207
both in the presence of infrared singularities and in their absence, by considering the two-loop radiative correction in QED to the process with three charged particles in the final state (as, for example, in the Bethe-Heitler process).
It's possible to consifer only diagrams with insertions of photon vertices into the external lines of the Born approximation diagrams, neglecting the photon momenta in the internal lines.
Note that according to the Landau criterion
[Landau:1959fi] L. D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys., 13(1):181–192, 1959.
the singularities of these diagrams are contained among the singularities of the total amplitude, since the latter include the singularities of diagrams in which some of the lines are missing.

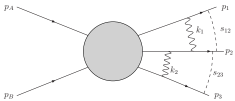
Let us denote the momenta of the final particles as p_1, p_2, p_3. For simplicity, we will assume that the masses of the particles are equal. Let

$$s_{ij} = (p_i + p_j)^2, \quad S = (p_1 + p_2 + p_3)^2 = s_{12} + s_{13} + s_{23} - 3m^2.$$

According to the accepted terminology, any two of three s_{ij} channels with $i < j$ are overlapping. One of these channels and the S-channel are non-overlapping, because the S-channel contains all 3 particles.

Let us investigate the presence of simultaneous discontinuities in the channels s_{12} and s_{23}. Such discontinuities can only be given by diagrams in which one of the photons connects the lines of particles with momenta p_1 and p_2, and the other with momenta p_3 and p_2.

One of this diagram is presented here.



The contribution of such a diagram is proportional to the Born amplitude multiplied by the factor

$$I \equiv I(s_{12}, s_{23}, s_{13}) = \int \frac{d^4 k_1}{(2\pi)^4 i} \frac{1}{d_{10} d_{11} d_{12}} J_3(\tilde{s}_{23}; \tilde{m}^2),$$

where $d_{10} = (k_1^2 - \lambda^2 + i0)$, $d_{11} = ((k_1 + p_1)^2 - m^2 + i0)$, $d_{12} = ((k_1 - p_2)^2 - m^2 + i0)$, $J_3(\tilde{s}_{23}); \tilde{m}^2)$ corresponds to the vertex with photon and electron virtualities s_{23} and $\tilde{m}^2)$ respectively, $\tilde{s}_{23} = (p_2 + p_3 - k_1)^2, \ \tilde{m}^2 = (p_2 - k_1)^2;$

$$J_3(\tilde{s}_{23};\tilde{m}^2) = \int \frac{d^4k_2}{(2\pi)^4 i}\frac{1}{d_{20}d_{21}d_{22}},$$

$$d_{20} = (k_2^2 - \lambda^2 + i0), \quad d_{21} = ((k_2+p_3)^2 - m^2 + i0),$$

$$d_{22} = ((k_1+k_1-p_2)^2 - m^2 + i0).$$

Let us consider the discontinuities of I with respect to the invariant s_{12}. There are only two such discontinuities: a two-particle one, which occurs due to the simultaneous vanishing of d_{11} and d_{12}, and a three-particle discontinuity, which occurs due to the simultaneous vanishing of d_{11}, d_{22}, and d_{20}. The second of them has no singularities, since the vanishing of any of the remaining denominators would contradict the law of conservation of energy-momentum. For a two-particle discontinuity, using the Cutkoski rules, we obtain

$$\Delta_{s_{12}} I = \int \frac{d^4k}{(2\pi)^4 i} \frac{(2\pi i)^2 \delta((p_1+k)^2 - m^2 \delta((p_2-k)^2 - m^2))}{k^2 - \lambda^2}$$

$$\times J_3(\tilde{s}_{23}).$$

Using the Sudakov parametrization (light-cone variables) with the light-cone vectors l_1 and l_2 such that

$$p_1 = l_1 + \frac{m^2}{\tilde{s}} l_2, \;\; p_2 = l_2 + \frac{m^2}{\tilde{s}} l_1, \;\; l_1^2 = 0, \;\; l_2^2 = 0, \;\; (l_1 + l_2)^2 = \tilde{s},$$

$$s_{12} = \tilde{s}(1 + \frac{m^2}{\tilde{s}})^2, \;\; \tilde{s} = s_{12}\frac{(1+v_{12})^2}{4}, \;\; v_{12} = \sqrt{1 - \frac{4m^2}{s}},$$

representing k as

$$k = -\beta l_1 + \alpha l_2 + k_\perp, \;\; (k_\perp l_1) = (k_\perp l_2) = 0, \;\; k_\perp^2 \equiv -\vec{k}^2 \le 0,$$

so that

$$k^2 = -\tilde{s}\alpha\beta - \vec{k}^2, \;\; (p_1+k)^2 - m^2 = \tilde{s}\alpha(1-\beta) - m^2\beta - \vec{k}^2,$$

$$(p_2-k)^2 - m^2 = \tilde{s}\beta(1-\alpha) - m^2\alpha - \vec{k}^2,$$

using $d^4k = \frac{\tilde{s}}{4} d\alpha d\beta d\vec{k}_\perp^2 d\phi$, passing to the integration variable $z = \beta/(1-m^2/\tilde{s})$,

for the s_{12} continuity we obtain

$$\Delta_{s_{12}} I = -\frac{2i}{(4\pi)^2\sqrt{s_{12}(s_{12}-4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} \int_0^{2\pi} d\phi J_3(\tilde{s}_{23}),$$

where ϕ is the azimuth angle of $\vec{k}_\perp$,

$$J_3(\tilde{s}_{23}) = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{ydy}{(y^2\tilde{p}_x^2 + \lambda^2(1-y))},$$

where $\tilde{p}_x = x(p_2-k) - (1-x)p_3$, and

$$(p_2-k)^2 = m^2, \;\; k = -z(p_1-p_2)+k_\perp, \;\; \vec{k}_\perp^2 = (s_{12}-4m^2)z(1-z).$$

Here it should be said that $\tilde{s}_{23} = (p_3 + p_2 - k)^2$ we must calculate in the physical region, since we use Sudakov's parametrization.

$$\tilde{p}_x^2 = m^2 - x(1-x)(s_{13}z + s_{23}(1-z) + 2(\vec{p}_{3\perp}\vec{k}_\perp)).$$

Let us consider the infrared singularities.
The most singular contribution to $\Delta_{s_{12}} I$ comes from $z = 0$ to $J((p_3 + p_2 - k)^2)$, so that for this contribution we have

$$\Delta_{s_{12}}^{sing} I = -\frac{4\pi i}{(4\pi)^2\sqrt{s_{12}(s_{12} - 4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12} - 4m^2}} J_3(s_{23}), .$$

Here $J_3(s_{23})$ is an usual vertex function. The existence of her cut makes obvious the existence of a double discontinuity in overlapping channels.

Noting that

$$-\frac{4\pi i}{(4\pi)^2\sqrt{s_{12}(s_{12}-4m^2)}}\int_0^1 \frac{dz}{z+\frac{\lambda^2}{s_{12}-4m^2}} = \Delta_{s_{12}}^{sing} J_3(s_{12}),$$

we obtain that the singular part of the double discontinuity is

$$\Delta_{s_{23}}\Delta_{s_{12}}^{sing} I = \Delta_{s_{12}} J_3(s_{12})\Delta_{s_{23}} J_3(s_{23})$$

according to infrared factorization.

It is worth noting that the double discontinuity (by s_{12} and s_{23}) may be not only from the contribution I under consideration, but also from the contribution I' corresponding to the diagram in which the vertices of the interaction of photons with a particle with momentum p_2 change places. But the calculation of the double discontinuity depends on the order in which the discontinuities are calculated. As already mentioned, $\Delta_{s_{23}} I$ has no singularities; similarly $\Delta_{s_{12}} I'$. Therefore $\Delta_{s_{23}}\Delta_{s_{12}}^{sing} I$ gives the full discontinuity.

Thus, infrared singularity destroys the hypothesis of the absence of simultaneous discontinuities in overlapping channels.

The representation for the discontinuity in the s_{12}-channel

$$\Delta_{s_{12}} I = -\frac{i}{(4\pi)\sqrt{s_{12}(s_{12}-4m^2)}}\int_0^1 \frac{dz}{z+\frac{\lambda^2}{s_{12}-4m^2}}\int_0^{2\pi}\frac{d\phi}{2\pi}J_3(\tilde{s}_{23})$$

can be used also for analysis of nonsigular contributions to double discontinuities. Using the Feynman parametrization

$$\frac{1}{d_{20}d_{21}d_{22}} = \int_0^1 dx \int_0^1 \frac{2y dy}{\left[(1-y)d_{20} + y\Big(x d_{22} + (1-x)d_{21}\Big)\right]^3}$$

and performing in $J_3(s_{23}, \tilde{m}^2)$ integration over d^4k_2, we obtain

$$J_3(\tilde{s}_{23}) = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{y dy}{(y^2\tilde{p}_x^2 + \lambda^2(1-y))},$$

where $\tilde{p}_x = x(p_2 - k) - (1 - x)p_3$,

$(p_2-k)^2 = m^2, \;\; k == -z(p_1-p_2)+k_\perp, \;\; \mathbf{k}_\perp^2 = (s_{12}-4m^2)z(1-z)$,

so that

$$\tilde{p}_x^2 = -x(1 - x)(p_3 + p_2 - k)^2 + m^2,$$
$$(p_3 + p_2 - k)^2 = s_{13}z + s_{23}(1 - z) + 2(\mathbf{p}_{3\perp}\mathbf{k}_\perp).$$

Performing integration over ϕ in the s_{12} - discontinuity $\Delta_{s_{12}} I$, one obtains

$$\Delta_{s_{12}} I = -\frac{i}{(4\pi)\sqrt{s_{12}(s_{12} - 4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} \overline{J_3},$$

where

$$\overline{J_3} = -\frac{1}{(4\pi)^2} \int_0^1 dx \int_0^1 \frac{ydy}{D},$$

$$D = \sqrt{\left(\lambda^2(1 - y) + y^2\left(m^2 - x(1 - x)\left(s_{23}(1 - z) + s_{13}z\right)\right)\right)^2 - B^2},$$

$$B^2 - 4(y^2x(1-x))^2\mathbf{p}_{3\perp}^2\mathbf{k}_\perp^2$$

$$= 4(y^2x(1-x))^2 z(1-z)\left[s_{23}s_{13} - \frac{m^2}{s_{12}}(s_{23}+s_{13}+s_{12}-4m^2)^2\right].$$

Let us consider the simplest case of a finite "photon mass" and zero electron mass.

In this case, the denominator becomes

$$D_0 \equiv D\big|_{m=0} = t\sqrt{t^2(s_{12}z - s_{23}(1-z))^2 + l^2 - 2lt(s_{12}z + s_{23}(1-z))},$$

where $t = y^2x(1-x), l = \lambda^2(1-y)$.

At negative s_{13} and s_{23}, as well as when s_{13} and s_{23} have different signs discontinuites in overlapping channels are absent, since the expression under the root is positive in the entire range of change of the integration variables x, y, z.

It is not so at positive s_{13} and s_{23}.

V.S. F., JETP Lett. **122** (2025) no.4, 203-207

Let us consider this case carefully.
By isolating the dimension from D, i.e. by making the substitution $D \to \lambda^2 D$ we obtain for D

$$D_0 = t\sqrt{(u-v):2+b^2-2b(u+v)}, u = \frac{s_{23}}{\lambda^2}(1-z), \quad v = \frac{s_{13}}{\lambda^2}z,$$

$$t = y^2x(1-x), \quad u_\pm = (\sqrt{b}\pm\sqrt{v})^2, b = \frac{(1-y)}{t}.$$

Thus, we obtain

$$\bar{J}_3 = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{ydy}{t}\frac{1}{\sqrt{(u-v)^2+b^2-2b(u+v)}}$$

It's convenient to take first integrals over x and y at fixed b.

$$\int_0^1 dx \int_0^1 \frac{ydy}{t} = \int_0^1 dx \int_0^1 \frac{ydy}{t} \int_0^\infty db\delta(b - \frac{(1-y)}{t})$$

$$= \int_0^\infty \frac{db}{b} \int_0^1 \frac{dy}{y} \int_0^1 dx\delta(x(1-x) - \frac{(1-y)}{by^2})$$

$$= \int_0^\infty \frac{db}{b} \int_0^1 \frac{dy}{y} \frac{2}{\sqrt{1 - 4\frac{(1-y)}{by^2}}} \theta(1-4\frac{(1-y)}{by^2}) = \int_0^\infty \frac{db}{b} \ln(1+b).$$

Note that no restrictions on b appeared here.

Thus,

$$\bar{J}_3 = -\frac{1}{(4\pi)^2}\int_0^\infty \frac{db}{b}\ln(1+b)\frac{1}{\sqrt{(u-v)^2+b^2-2b(u+v)}}.$$

Let us for definiteness start with $v-u>0$. Using

$$\int \frac{db}{b\sqrt{(u-v)^2+b^2-2b(u+v)}} = \frac{1}{v-u}\mathcal{L},$$

where

$$\mathcal{L} = \ln\left(\frac{(\sqrt{(v-u)^2+b^2-2b(v+u)}-b-(v-u))(u)}{(\sqrt{(v-u)^2+b^2-2b(v+u)}-b+(v-u))(v)}\right),$$

with the argument of $\mathcal{L}$ tending to 1 at $b\to\infty$,
we perform integration by parts and obtain

$$\bar{J}_3 = -\frac{1}{(4\pi)^2}\int_0^\infty \frac{db}{(1+b)(v-u)}\mathcal{L}$$

Tending of the argument of $\mathcal{L}$ to 1 at $b \to \infty$ ensures the absence of an extra-integral term and the convergence of the resulting integral at infinity. The argument of $\mathcal{L}$ varies for $v > 0,\ u < 0$ from zero for $b = 0$ to 1 for $b \to \infty$, remaining positive. It is worth noting that for $v > 0,\ u < 0$, the expression under the radical in $\sqrt{(v-u)^2 + b^2 - 2b(v+u))}$ is positive, so the root extracted from it should also be considered positive.
To rationalize the square root we introduce the variable t as

$$\sqrt{(v-u)^2 + b^2 - 2b(v+u))} = t + b,$$

with

$$b = \frac{(v-u)^2 - t^2}{2(t+v+u)};$$

$$\frac{db}{1+b} = dt\left[\frac{2(1-t)}{2(t+v+u) + (v-u)^2 - t^2} - \frac{1}{t+v+u}\right],$$

t changes from $v-u$ (at $b = 0$)to $-(v+u)$ (at $b = \infty$), so that

$$\bar{J}_3 = -\frac{1}{(4\pi)^2(v-u)}\int_{-(v+u)}^{v-u} dt\left[\frac{2(1-t)}{2(t+v+u)+(v-u)^2-t^2}\right.$$
$$\left.-\frac{1}{t+v+u}\right]\ln\frac{(v-u-t)(-u)}{(v-u+t)(v)}.$$

Integrating by parts again, we obtain a fairly simple representation

$$\bar{J}_3 = -\frac{1}{(4\pi)^2(v-u)}\int_{\tau_-}^{\tau_+} dt\left[\frac{1}{(\tau_+-t)}+\frac{1}{(t+\tau_+)}\right]\ln\frac{(t_+-t)(t-t_-)}{2(t-\tau_-)},$$

where

$$\tau_- = -(v+u),\ \ \tau_+ = v-u,\ \ t_\pm = \pm\sqrt{(1+v+u)^2-4vu}+1$$
$$= 1\pm\sqrt{1+2(v+u)+(v-u)^2}.$$

Recall that we are in the region $v > 0$.

Let's consider how the integral changes with changing u. For $u < 0$ we have

$$t_- < \tau_- < \tau_+ < t_+,$$

the entire domain of integration lies within the interval $[t_-, t_+]$, so the argument of the logarithm is positive in the domain of integration. The singularity of the first term in square brackets is canceled by the logarithm, and the singularity of the second lies below the domain of integration, so the integral is finite and real. For $u = 0$

$$t_- = \tau_- < \tau_+ < t_+,$$

the integral takes the form

$$\bar{J}_3 = -\frac{1}{(4\pi)^2 v} \int_{-v}^{v} dt \left[\frac{1}{(v-t)} + \frac{1}{(t+v)} \right] \ln(1 + \frac{v-t}{2}).$$

Here, as before, the singularity of the first term in square brackets is canceled out by the logarithm, but the singularity of the second lies on the boundary of the integration domain, so that the integral acquires a logarithmic singularity. At $u > 0$ we have

$$\tau_- < t_- < -\tau_+ < \tau_+ < t_+,$$

The singularity of the first term in square brackets is still canceled by the logarithm. The most important circumstance is that in the interval $[\tau_-, t_-]$ the argument of the logarithm becomes negative, that is, the logarithm (and therefore the entire integral) acquires an imaginary part.

$$\Im \bar{J}_3 = -\frac{1}{(4\pi)^2(v-u)} \int_{\tau_-}^{t_-} dt \left[\frac{-1}{(t-(v-u))} + \frac{1}{(t+(v-u))} \right] \Im \ln(t - t_-)$$

Moreover, for $u < v$, the singularity of the second term is in the integration domain. However, it lies for $t > t_-$, where the logarithm's argument is real, and is also canceled by the logarithm.

Thus, for $v > 0$, INT has a cut at u when $u > 0$. On the upper bank of the cut, $\Im \ln((t - t_-) = -\pi$ (this is easier to see by moving to the integration variable $x = t + v + u$ with integration limits independent of u), so that

$$\Im \bar{J}_3 = -\frac{\pi}{(4\pi)^2(v-u)} \ln \frac{(t_- - (v-u))(\tau_- + (v-u))}{(t_- + (v-u))(\tau_- - (v-u))}$$

$$= -\frac{\pi}{(4\pi)^2(v-u)} \ln \frac{1 + \sqrt{(1+v+u)^2 - 4uv} + v - u}{1 + \sqrt{(1+v+u)^2 - 4uv} - (v-u)}.$$

This result can be obtained in another way, using that

$$\Im \bar{J}_3 = -\frac{1}{(4\pi)^2} \int_0^\infty \frac{db}{b} \ln(1+b) \Im \frac{1}{\sqrt{(u-v)^2 + b^2 - 2b(u+v)}}$$

$$-\frac{1}{(4\pi)^2} \frac{1}{2i} \oint_C \frac{dz}{z} \ln(1+z) \Im \frac{1}{\sqrt{(u-v)^2 + z^2 - 2z(u+v)}},$$

where the contour C surrounds the cut of the square root.

Then deforming the contour so that it surrounds the cut of $\ln(1+z)$, we obtain

$$\Im \bar{J}_3 = -\frac{\pi}{(4\pi)^2}\int_{-\infty}^{-1}\frac{db}{b}\frac{1}{\sqrt{(u-v)^2+b^2-2b(u+v)}},$$

which gives the desired result.

Thus, the double discontinuity in the overlapping channels s_{12} and s_{23} is

$$\Delta_{s_{23}}\Delta_{s_{12}} I = \frac{i}{(4\pi)}\int_0^1 \frac{dz}{s_{12}z+\lambda^2}$$

$$\times\frac{i}{(8\pi)(s_{13}z-s_{23}(1-z))}\ln\frac{1+\sqrt{(1+v+u)^2-4uv}+v-u}{1+\sqrt{(1+v+u)^2-4uv}-(v-u)},$$

where

$$u=\frac{s_{23}}{\lambda^2}(1-z),\quad v=\frac{s_{13}}{\lambda^2}z.$$

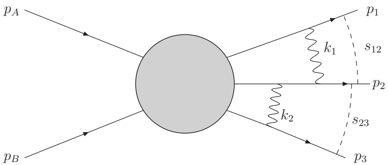
Thus, we see that the statement about the absence of simultaneous discontinuities in energy invariants of overlapping channels contradicts not only well known factorization infrared singularities, but is not correct as well in the case of absence of such singularities.

Actually, in both cases we have shown the existence of not only simultaneous but also double discontinuities.
The question arise: what is the reason of the contradiction with usually accepted statement? And what does Steinmann's relationship have to do with this statement?

Initially, this statement arose in Regge theory as a convenient assumption. The Steinmann relations were mentioned in connection with this assumption, but there was a clear understanding that they could not serve as a justification for this assumption, since they were obtained for retarded commutators, and not for *S*-matrix elements.

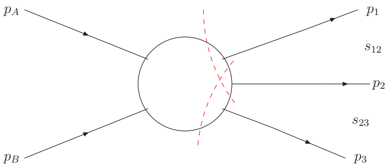
But now the Steinmann relations are used as evidence of the ststement.
In my opinion, a substitution was carried out in
H. P. Stapp, Phys. Rev. D 3, 3177 (1971),
where from expressions of S- matrix elements in terms of vacuum averages from time-ordered products of Heisenberg field operators it is concluded that there are no simultaneous discontinuities of S-matrix elements by the energies of overlapping channels, and it is claimed that these are the Steinmann relations (that absolutely untrue). At the same time, the conclusion about the absence of simultaneous discontinuities of S-marix elements does not stand up to criticism, if only because the number of energy denominators in formula is equal to their number in the Born approximation.

Actually, the fact that in the time ordered perturbation theory the energy denominators with the energies of overlapping channels do not occur simultaneously does not mean the absence of simultaneous discontinuities. In the example we are considering



it means that at any time ordering the energy denominators $E_1 + E_2 - \sum_i E_i$ and $E_2 + E_3 - \sum_j E_j$, where E_i and E_j are energies of some intermediate particles, can not be simultaneously. But, as we see, it does not forbid double discontinuities in the s_{12} and s_{12} channels.

Another argument in favour of absence of simultaneous discontinuities is based on the picture



Summary

- The statement of absence of simultaneous discontinuities in energy invariants of overlapping channels of multiparticle amplitudes is not correct.
- An obvious refutation of this statement is the factorization of infrared singularities.
- Simultaneous discontinuities exist in the absence of infrared singularities as well.
- Steinmann relations have no relations to S-matrix and can not be used for justification of this statement.
- Existence of of simultaneous discontinuities means that the commonly accepted Regge form of the multiparticle amplitudes is not valid in QCD.
- It must be taken into account in dispersive derivation of the BFKL equation in higher approximations.