Correlator of heavy–light quark currents in HQET in the large β_0 limit

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HQET

$$S_0(x) = \delta(\vec{x}\,) S_0(x^0) \qquad S_0(t) = -i\theta(t) \qquad S_0(p) = \frac{1}{p^0 + i0}$$

no loops

Heavy–light quark currents

$$j_{P0} = \frac{1 + P\gamma_0}{2} \varphi_0^* q_0 \qquad P = \pm 1$$

Correlators

$$\langle T j_{P0}(x) \bar{\jmath}_{P0}(0) \rangle = \delta(\vec{x}\,) P \frac{1 + P\gamma_0}{2} \Pi_{P0}(x^0)$$

Analytically continue $\Pi_{P0}(t)$ from $t > 0$ to $t = -i\tau$, $\tau > 0$:

$$\Pi_{P0}(\tau)$$

Operator product expansion

$$\Pi_P(\omega,\mu) = \sum_{\mathcal{O}} C_{P,\mathcal{O}}(\omega,\mu)\langle\mathcal{O}(\mu)\rangle$$

$$\Pi_P(\tau,\mu) = \sum_{\mathcal{O}} C_{P,\mathcal{O}}(\tau,\mu)\langle\mathcal{O}(\mu)\rangle$$

$\Pi_P(\omega,\mu)$ contains $1/\varepsilon^n$ UV poles polynomial in ω
$\Pi_P(t,\mu)$ contains $1/\varepsilon^n$ UV poles $\delta^{(n)}(t)$
$\Pi_P(\tau,\mu)$ contains no $1/\varepsilon^n$ poles
$\rho_P(\omega,\mu)$ contains no $1/\varepsilon^n$ poles

Operators with $D \le 2$

Perturbative contributions

$$\mathcal{O}_0 = 1\,, \quad m_0 \quad m_0^2 \quad \sum m_{i0}^2$$

m — mass of q in j
m_i — masses of all light flavors

- Even D — Π_P contains no P
- Odd D — $\Pi_P \propto P$

Assume $P = +1$

History

- 2 loops: Broadhurst, Grozin (1992); Bagan, Ball, Braun, Dosch (1992); Neubert (1992)
- 3 loops: Chetyrkin, Grozin (2022)
- 4 loops: Grozin (2024)

Large β_0 limit

$n \in [0,2]$ $\qquad C^{(1)}_{m^n,0}(\tau)$ is finite

$C_{m^n,0}(\omega) \Rightarrow C_{m^n,0}(t) \Rightarrow C_{m^n,0}(\tau)$, same for 1 loop

$$A_{n0}(\tau) = \frac{C_{m^n,0}(\tau)}{C^{(1)}_{m^n,0}(\tau)} = 1 + \sum_{l=1}^{\infty} \sum_{k=0}^{L-1} a'_{nlk} n_f^k \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^l$$

$$= 1 + \sum_{l=1}^{\infty} \sum_{k=0}^{L-1} a_{nlk} \beta_0^k \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^l$$

$\beta_0 \alpha_s \sim 1$ $\qquad 1/\beta_0$ — small parameter

$$A_{n0} = 1 + \frac{C_F}{\beta_0} \sum_{l=1}^{\infty} \frac{F_n(\varepsilon, l\varepsilon)}{l} \left[\frac{\beta_0 g_0^2}{(4\pi)^{d/2}} \left(\frac{\tau e^{\gamma}}{2} \right)^{2\varepsilon} e^{-\gamma\varepsilon} \frac{D(\varepsilon)}{\varepsilon} \right]^l$$

$$+ \mathcal{O}\left(\frac{1}{\beta_0^2} \right)$$

$$D(\varepsilon) = 6 e^{\gamma\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(2-\varepsilon)}{\Gamma(4-2\varepsilon)} = 1 + \frac{5}{3}\varepsilon + \cdots$$

$$b = \beta_0 \frac{\alpha_s(\mu)}{4\pi} \sim 1$$

$$A_{n0} = 1 + \frac{C_F}{\beta_0} \sum_{l=1}^{\infty} \frac{F_n(\varepsilon, l\varepsilon)}{l} \left[\frac{b}{\varepsilon + b} \left(\frac{\mu \tau e^{\gamma}}{2} \right)^{2\varepsilon} D(\varepsilon) \right]^l + \mathcal{O}\left(\frac{1}{\beta_0^2} \right)$$

$$\mu_\tau = \frac{2e^{-\gamma}}{\tau} D(\varepsilon)^{-1/(2\varepsilon)} \to \frac{2}{\tau} e^{-\gamma - 5/6}$$

$$A_{n0} = 1 + \frac{C_F}{\beta_0} \sum_{l=1}^{\infty} \frac{F_n(\varepsilon, l\varepsilon)}{l} \left(\frac{b}{\varepsilon + b} \right)^l + \mathcal{O}\left(\frac{1}{\beta_0^2} \right)$$

$$F_n(\varepsilon, u) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} F_{n,ij} \varepsilon^i u^j$$

Expand $F_n(\varepsilon, u)$ in ε, u and $(b/(\varepsilon + b))^l$ in b
— quadruple sum

Anomalous dimension

Collecting ε^{-1} terms: only $F_{n,i0}$
higher ε^{-n} are fixed by finiteness of γ

$$\gamma_n = 2\gamma_j - n\gamma_m \qquad n \in [0,2]$$

$$\gamma_n(b) = -2C_F \frac{b}{\beta_0} F_n(-b,0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_j(b) = -\frac{1}{2}\gamma_m(b)$$

$$= -C_F \frac{b}{\beta_0} \frac{1+\frac{2}{3}b}{B(2+b,2+b)\Gamma(3+b)\Gamma(1-b)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_n(b) = 2(n+1)\gamma_j(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_{n0} = -2C_F F_n(0,0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \qquad F_n(0,0) = 3(n+1)$$

Renormalization group

$$A_n(\tau,\mu) = \hat{A}_n \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_\tau)}\right)^{\gamma_{n0}/(2\beta_0)} K_n(\alpha_s(\mu))$$

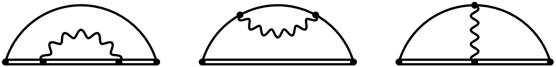
$$K_n(\alpha_s) = \exp\int_0^{\alpha_s} \frac{d\alpha_s}{\alpha_s}\left(\frac{\gamma_n(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_{n0}}{2\beta_0}\right)$$

Collecting ε^0 terms: only $F_{n,i0}$ and $F_{n,0j}$

$$\hat{A}_n = 1 + \frac{C_F}{\beta_0}\int_0^\infty du\, e^{-u/b} S_n(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

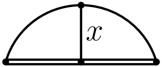
$$S_n(u) = \frac{F_n(0,u) - F_n(0,0)}{u} \qquad b = b(\mu_\tau)$$

Calculation



Gluon denominator raised to $1 + u - \varepsilon$

$F(\varepsilon, u)$ via $I(1 + u - \varepsilon)$

$I(x) =$ 

expressible via ${}_3F_2(1)$: Beneke, Braun (1994)

$$S_0(u) = \frac{1}{u}\left[\frac{2(3-5u+u^2)\Gamma(1-u)e^{-2\gamma u}}{(1-u)(2-u)(1-2u)\Gamma(1+u)} - 3\right]$$
$$+ \frac{4u(1-2u)e^{-2\gamma u}}{\Gamma(1+2u)} I$$
$$I = \frac{\Gamma(1-u)}{u^3}\left[\Gamma(1-u)\Gamma^2(1+u) - \frac{\Gamma^2(1+2u)}{\Gamma(1+3u)} F(u)\right]$$
$$F(u) = {}_3F_2\left(\begin{array}{c} u, u, 2u \\ 1+u, 1+3u \end{array} \middle| 1\right)$$
$$S_n(u) = \sum_{k=0}^{\infty} s_{nk} u^k$$
$$\hat{A}_n = 1 + C_F \frac{b}{\beta_0} \sum_{l=1}^{\infty} (l-1)!\, s_{n,l-1} b^{l-1} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Renormalons

Consider $n = 0$

$$\hat{A} = 1 + \frac{C_F}{\beta_0} \int_0^\infty du\, e^{-u/b} S(u)$$

Pole

$$S \sim \frac{r}{u_0 - u} \qquad u_0 > 0$$

Residue $\Rightarrow$ renormalon ambiguity

$$\Delta A = \frac{C_F}{\beta_0} r \left(e^{\gamma + 5/6} \Lambda_{\overline{\mathrm{MS}}} \frac{\tau}{2} \right)^{2u_0}$$

Asymptotic series. $\Delta A \sim$ the minimum term in the series. Prescription, e.g. principal value. Can be changed, e.g., cut out $[u_0 - \delta, u_0 + 2\delta]$.

UV renormalon $u = \frac{1}{2}$

$$S(u) \sim e^{-\gamma} \frac{4}{\frac{1}{2} - u}$$

$$\frac{\Delta\Pi(\tau)}{\Pi(\tau)} = 2\frac{C_F}{\beta_0} e^{5/6} \Lambda_{\overline{\mathrm{MS}}} \tau = -\Delta\bar{\Lambda}\,\tau$$

$$\Delta\bar{\Lambda} = -2\frac{C_F}{\beta_0} e^{5/6} \Lambda_{\overline{\mathrm{MS}}}$$

$$\tau \to \infty : \quad \Pi(\tau) \propto e^{-\bar{\Lambda}\tau}$$

Beneke, Braun (1994)

IR renormalons $u = 3, 4, 5, \ldots$

$u = 1, 2$ — no poles

$$S(u) \sim -e^{-6\gamma_E} \left(\frac{1}{36} \frac{1}{(3-u)^2} + \frac{2}{27} \frac{1}{3-u} + \cdots \right)$$

$$\frac{\Delta\Pi(\tau)}{\Pi(\tau)} = -\frac{1}{864} \frac{C_F}{\beta_0} \left(e^{5/6} \Lambda_{\overline{\mathrm{MS}}} \tau \right)^6$$

The IR renormalon ambiguity in C_1 is compensated by UV renormalon ambiguities of $\langle (\bar{q}q)^2 \rangle$ Beneke, Braun (1994). There are no appropriate operators to compensate $u = 1$ and 2 poles ($\langle G^2 \rangle$ does not contribute to $\Pi(\tau)$ at 1 loop).

$A_{1,2}$

- UV $u = \frac{1}{2}$
- IR $u = 1,\, 2,\, 3\, \ldots$

$$S_1(u) \sim e^{-2\gamma_E} \frac{6}{1-u} - e^{-4\gamma_E} \left(\frac{3}{2} \frac{1}{(2-u)^2} + \frac{9}{2} \frac{1}{2-u} \right) + \cdots$$

$$S_2(u) \sim e^{-2\gamma_E} \frac{3}{1-u} - e^{-4\gamma_E} \left(\frac{1}{2} \frac{1}{(2-u)^2} + \frac{9}{4} \frac{1}{2-u} \right) + \cdots$$

$\sum m_i^2$

$$C^0_{\sum m_i^2}(\tau) = -\frac{16}{3}\frac{N_c C_F T_F}{(4\pi)^{d/2}}\left(\frac{2}{\tau}\right)^{1-2\varepsilon}\left\{\frac{1}{\beta_0^2}\sum_{l=1}^{\infty}\frac{F_\Sigma(\varepsilon,l\varepsilon)}{l}\right.$$

$$\left.\times\left[\frac{\beta_0 g_0^2}{(4\pi)^{d/2}}\left(\frac{\tau e^\gamma}{2}\right)^{2\varepsilon}e^{-\gamma\varepsilon}\frac{D(\varepsilon)}{\varepsilon}\right]^l + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)\right\}$$

$$= -\frac{16}{3}\frac{N_c C_F T_F}{(4\pi)^{d/2}}\left(\frac{2}{\tau}\right)^{1-2\varepsilon}\left[\frac{1}{\beta_0^2}\sum_{l=1}^{\infty}\frac{F_\Sigma(\varepsilon,l\varepsilon)}{l}\left(\frac{b}{\varepsilon+b}\right)^l + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)\right]$$

$(\mu = \mu_\tau)$ $\quad$ $F_\Sigma(\varepsilon,\varepsilon) = F_\Sigma(\varepsilon,0) = 0$ $\quad$ No $l = 1$; no ε^{-n}

$$C_{\Sigma m_i^2}(\tau) = -\frac{2N_c C_F T_F}{3\pi^2\tau}\left[\frac{1}{\beta_0^2}\int_0^\infty du\, e^{-u/b}S_\Sigma(u) + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)\right]$$

$$S_\Sigma(u) = \frac{F_\Sigma(0,u)}{u}$$

IR renormalons $u = 2, \ldots$

Spectral density

$$\rho_P(\omega,\mu) = \sum_{\mathcal{O}} R_{P,\mathcal{O}}(\omega,\mu)\langle\mathcal{O}(\mu)\rangle$$

$$\tilde{A}_{n0}(\omega) = \frac{R_{m^n,0}(\omega)}{R^{(1)}_{m^n,0}(\omega)} =$$

$$1 + \frac{C_F}{\beta_0}\sum_{l=1}^{\infty}\frac{\tilde{F}_n(\varepsilon,l\varepsilon)}{l}\left[\frac{b}{\varepsilon+b}\left(\frac{\mu}{2\omega}\right)^{2\varepsilon}D(\varepsilon)\right]^l + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\mu_\omega = 2\omega D(\varepsilon)^{-1/(2\varepsilon)} \to 2\omega e^{-5/6}$$

$$\tilde{A}_n = 1 + \frac{C_F}{\beta_0}\sum_{l=1}^{\infty}\frac{\tilde{F}_n(\varepsilon,l\varepsilon)}{l}\left(\frac{b}{\varepsilon+b}\right)^l + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\tilde{F}_n(\varepsilon,u) = \frac{\Gamma(3-n-2\varepsilon)}{\Gamma(3-n-2u-2\varepsilon)}e^{2\gamma u}F_n(\varepsilon,u)$$

$\tilde{A}_0$

UV renormalon pole $u = \frac{1}{2}$

$$\frac{\Delta\rho(\omega)}{\rho(\omega)} = 2\frac{\Delta\bar{\Lambda}}{\omega} \qquad \rho(\omega) \propto \omega^2 \qquad \Delta\omega = \Delta\bar{\Lambda}$$

IR renormalon poles $u = 3, \ldots$

$$\tilde{S}_0(u) \sim \frac{2}{3}\frac{1}{3-u} \qquad \frac{\Delta\rho(\omega)}{\rho(\omega)} = \frac{1}{96}\frac{C_F}{\beta_0}\left(\frac{e^{5/6}\Lambda_{\overline{\text{MS}}}}{\omega}\right)^6$$

Compensated by UV renormalon ambiguity of $\langle(\bar{q}q)^2\rangle$

$\tilde{A}_{1,2}$

IR renormalon poles $u = 2, \ldots u = 1$ simple pole disappears due to $F_n \Rightarrow \tilde{F}_n$, double poles $u = 2, \ldots$ become simple

$$\tilde{S}_1(\omega) \sim -\frac{6}{2-u} + \cdots \qquad \tilde{S}_2(\omega) \sim \frac{6}{2-u} + \cdots$$

$$\frac{\Delta R_m(\omega)}{R_m(\omega)} = -\frac{3}{2}\frac{C_F}{\beta_0}\left(e^{5/6}\frac{\Lambda_{\overline{\rm MS}}}{\omega}\right)^4$$

$$\frac{\Delta R_{m^2}(\omega)}{R_{m^2}(\omega)} = -3\frac{C_F}{\beta_0}\left(e^{5/6}\frac{\Lambda_{\overline{\rm MS}}}{\omega}\right)^4$$

- $R_m(\omega)m$: compensated by the UV renormalon ambiguity of the vacuum average in $m\langle\bar{q}G\sigma q\rangle$
- $R_{m^2}m^2$: compensated by the UV renormalon ambiguity of the gluon condensate in $m^2\langle G^2\rangle$

$\sum m_i^2$

$$\tilde{F}_\Sigma(\varepsilon, u) = \frac{e^{2\gamma_E u}}{\Gamma(1 - 2\varepsilon - 2u)} F_\Sigma(\varepsilon, u)$$

$\tilde{S}_\Sigma(u)$: IR renormalon poles $u = 2, \ldots$