Dispersive approach to the hadronic light-by-light contribution to $(g-2)_\mu$

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FOR FUNDAMENTAL PHYSICS

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Based on:

JHEP09(14)091, JHEP09(15)074, JHEP04(17)161, PRL(17)
in collab. with M. Hoferichter, M. Procura and P. Stoffer and
PLB738(2014)6 .... +B. Kubis
and work in progress with F. Hagelstein and L. Laub

Outline

Introduction

Setting up the stage: Master Formula

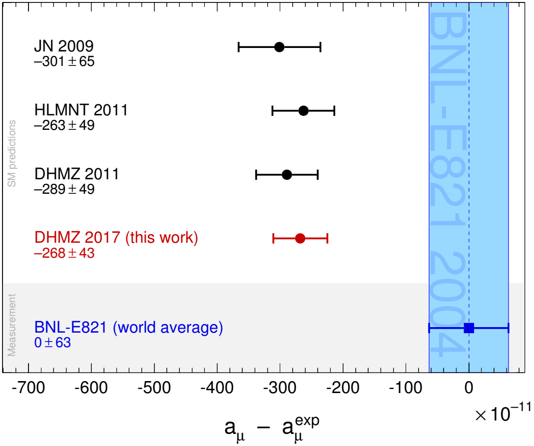
A dispersion relation for HLbL
- Pion-pole contribution
- Pion-box contribution
- Pion rescattering contribution

Short-distance constraints

Outlook and Conclusions

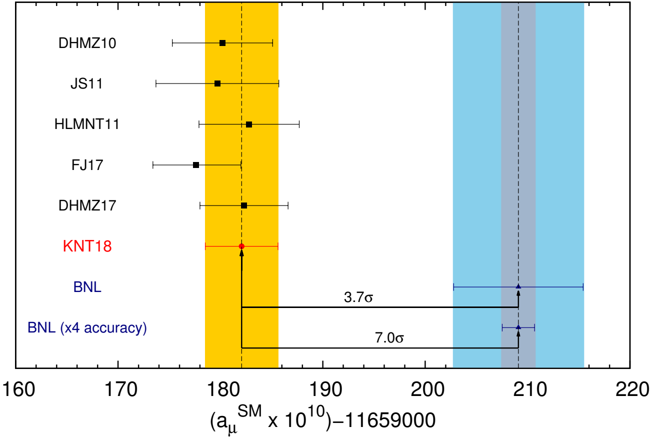
Status of $(g-2)_\mu$, experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2017



Status of $(g-2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)



Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon "$(g-2)$ Theory Initiative" led by A. El-Khadra and C. Lehner

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.97	0.07
electroweak, total	153.6	1.0
HVP (LO) [KNT 18]	6 932.7	24.6
HVP (NLO) [KNT 18]	−98.2	0.4
HLbL [update of Glasgow consensus–KNT 18]	98.0	26.0
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.0	2.0
theory	116 591 820.5	35.6

Status of $(g-2)_\mu$, experiment vs SM

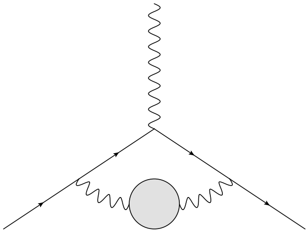
KNT 18

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 268.5 \pm 72.4 \qquad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved



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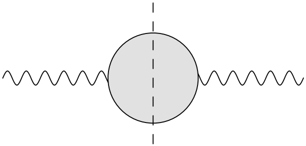
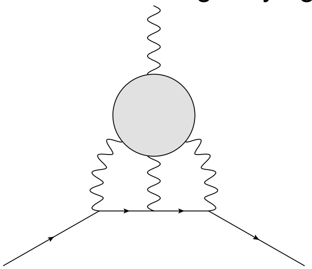


- basic principles: unitarity and analyticity
- direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
- alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved
- Hadronic light-by-light (HLbL) is more problematic:



- 4-point fct. of em currents in QCD
- "*it cannot be expressed in terms of measurable quantities*"
- until recently, only model calculations
- lattice QCD is making fast progress

Different analytic evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
" " + subl. in N_c	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*, see below)
- heavier single-particle poles decreasingly important

Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible
  (in principle)
- if data not available: use theoretical calculations of subamplitudes, short-distance constraints etc.

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- if data not available: use theoretical calculations of subamplitudes, short-distance constraints etc.
- First attempts: GC, Hoferichter, Procura, Stoffer (14)
  Pauk, Vanderhaeghen (14)
- similar philosophy, with a different implementation:
  Schwinger sum rule Hagelstein, Pascalutsa (17)
- why hasn't this been adopted before?

Outline

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz \, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{ j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \dots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \dots\}$, but in $d = 4$ only 136 are linearly independent Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

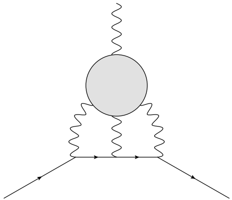
- 43 basis tensors (BT) — in $d = 4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

Master Formula

$$a_\mu^{\mathrm{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- the Π_i are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques



GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

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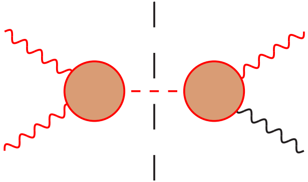
$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
Input: pion transition form factor Hoferichter et al. (18)
First results of direct lattice calculations Gerardin, Meyer, Nyffeler (16)

Setting up the dispersive calculation

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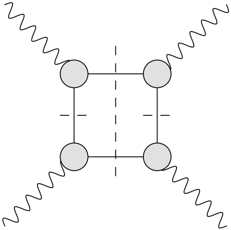
π-box with the BTT set:



– we have constructed a Mandelstam representation for the contribution of the 2-pion cut with LHC due to a pion pole
– we have explicitly checked that this is identical to sQED multiplied by $F_V^\pi(s)$ (FsQED)

Setting up the dispersive calculation

We split the HLbL tensor as follows:

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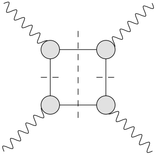
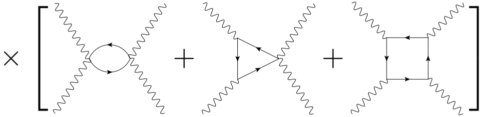


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$



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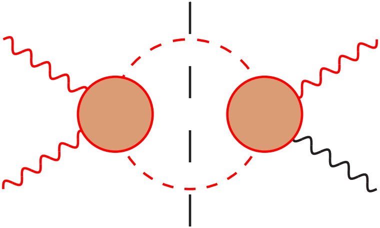

The "rest" with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

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E.g. $\gamma^*\gamma^* \to \pi\pi$ S-wave contributions

$$\hat{\Pi}^S_4 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{-2}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(4s' \,\mathrm{Im} h^0_{++,++}(s') - (s'+q_1^2-q_2^2)(s'-q_1^2+q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

$$\hat{\Pi}^S_5 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{-2}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(4t' \,\mathrm{Im} h^0_{++,++}(t') - (t'+q_1^2-q_3^2)(t'-q_1^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}^S_6 = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{-2}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(4u' \,\mathrm{Im} h^0_{++,++}(u') - (u'+q_2^2-q_3^2)(u'-q_2^2+q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}^S_{11} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty du' \frac{4}{\lambda_{23}(u')(u'-q_1^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(u') - (u'-q_2^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(u')\Big)$$

$$\hat{\Pi}^S_{16} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{4}{\lambda_{13}(t')(t'-q_2^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(t') - (t'-q_1^2-q_3^2)\,\mathrm{Im} h^0_{00,++}(t')\Big)$$

$$\hat{\Pi}^S_{17} = \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{4}{\lambda_{12}(s')(s'-q_3^2)^2}\Big(2\,\mathrm{Im} h^0_{++,++}(s') - (s'-q_1^2-q_2^2)\,\mathrm{Im} h^0_{00,++}(s')\Big)$$

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We split the HLbL tensor as follows:

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Contributions of cuts with anything else other than one and two pions in intermediate states are neglected in first approximation

of course, the η, η' and other pseudoscalars pole contribution, or the kaon-box/rescattering contribution can be calculated within the same formalism

Pion-pole contribution

- Expression of this contribution in terms of the pion transition form factor already known Knecht-Nyffeler (01)
- Both transition form factors (TFF) must be included:

$$\bar{\Pi}_1 = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0)}{q_3^2 - M_{\pi^0}^2}$$

[dropping one bc short-distance not correct Melnikov-Vainshtein (04)]

- data on singly-virtual TFF available CELLO, CLEO, BaBar, Belle, BESIII
- several calculations of the transition form factors in the literature Masjuan & Sanchez-Puertas (17), Eichmann et al. (17), Guevara et al. (18)
- dispersive approach works here too Hoferichter et al. (18)
- quantity where lattice calculations can have a significant impact Gerardin, Meyer, Nyffeler (16)

Pion-pole contribution

Latest complete analyses:

- Dispersive calculation of the pion TFF — Hoferichter et al. (18)

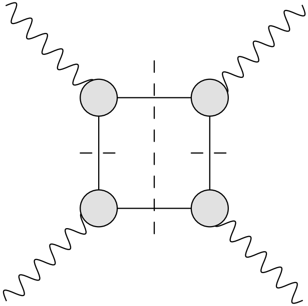
$$10^{11} a_\mu^{\pi^0} = 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\left({}^{2.2}_{1.4}\right)_{\text{BL}}(0.5)_{\text{asym}} = 62.6^{+3.0}_{-2.5}$$

- Padé-Canterbury approximants — Masjuan & Sanchez-Puertas (17)

$$10^{11} a_\mu^{\pi^0} = 63.6(1.3)_{\text{stat}}(0.6)_{a_{P;1,1}}(2.3)_{\text{sys}} = 63.6(2.7)$$

Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

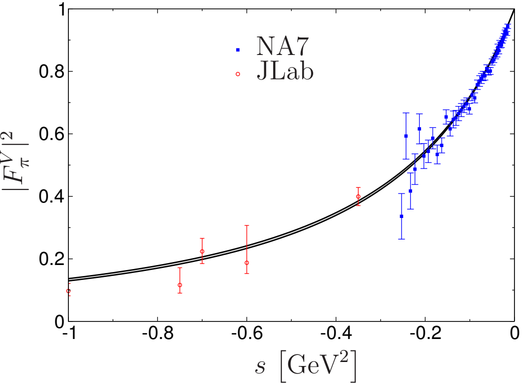
where

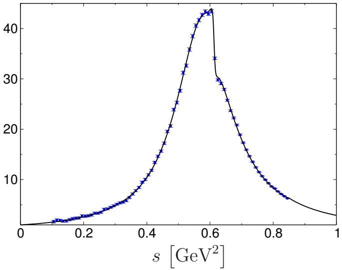
$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion-box contribution





Uncertainties are negligibly small:

$$a_\mu^{\mathrm{FsQED}} = -15.9(2) \cdot 10^{-11}$$

Pion-box contribution

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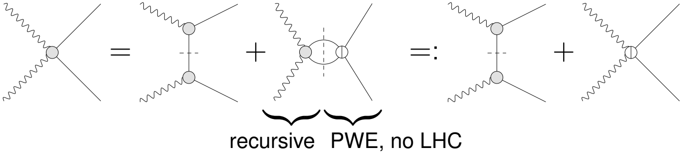
First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:



Based on:

- taking the pion pole as the only left-hand singularity
- $\Rightarrow$ pion vector FF to describe the off-shell behaviour
- $\pi\pi$ phases obtained with the inverse amplitude method
  [realistic only below 1 Gev: accounts for the $f_0(500)$ + unique and well defined extrapolation to ∞]
- numerical solution of the $\gamma^*\gamma^* \to \pi\pi$ dispersion relation

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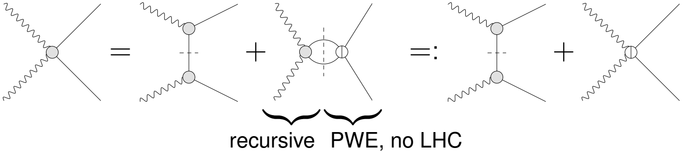

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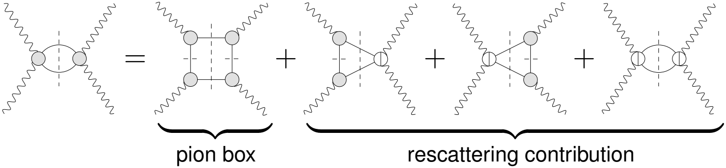
S-wave contributions : $\quad a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$

Two-pion contribution to $(g-2)_\mu$ from HLbL

Two-pion contributions to HLbL:



$$\boxed{a_\mu^{\pi-\mathrm{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1)\cdot 10^{-11}}$$

$\gamma^*\gamma^* \to \pi\pi$ contribution from other partial waves

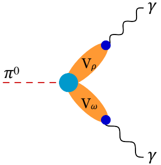
- formulae get significantly more involved with several subtleties in the calculation
- in particular sum rules which link different partial waves must be satisfied by different resonances in the narrow width approximation Danilkin, Pascalutsa, Pauk, Vanderhaeghen (12,14,17)
- data and dispersive treatments available for on-shell photons e.g. Dai & Pennington (14,16,17)
- dispersive treatment for the singly-virtual case and check with forthcoming data is very important $\to$ talks by Prencipe, & Redmer

Outline

Short-distance contraints

- short-distance constraints on *n*-point functions in QCD is a well known issue
- low- and intermediate-energy representation in terms of hadronic states doesn't typically extrapolate to the right high-energy limit
- requiring that the latter be satisfied is often essential to obtain a description of spectral functions which leads to correct integrals over them
  vast literature [de Rafael, Goltermann, Peris,...]
- implementing such an approach for HLbL not very simple, but it works
  GC, Hagelstein, Laub, work in progress

A Regge-like large-N_C inspired model



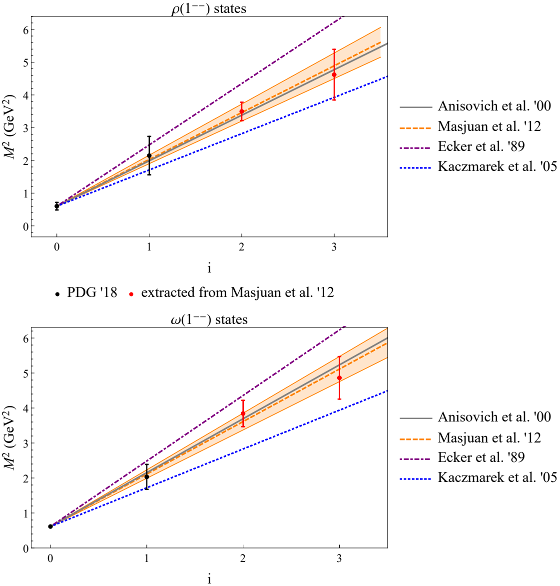
$$F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = \sum_{V_\rho, V_\omega} \frac{F_{V_\rho}(q_1^2)\, F_{V_\omega}(q_2^2)\, G_{\pi V_\rho V_\omega}(q_1^2, q_2^2)}{(q_1^2 + M_{V_\rho}^2)(q_2^2 + M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

where

$$M_{V_{\rho,\omega}}^2 = M_{\rho,\omega}^2(i_{\rho,\omega}) = M_{\rho,\omega}^2(0) + i_{\rho,\omega}\, \sigma_{\rho,\omega}^2$$

Masjuan, Broniowski, Ruiz Arriola (12)

A Regge-like large-N_C inspired model



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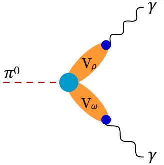

$$F_{\pi^{(n)}\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho,V_\omega} \frac{F_{V_\rho}(q_1^2)\,F_{V_\omega}(q_2^2)\,G_{\pi^{(n)}V_\rho V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

where

$$M_{V_{\rho,\omega}}^2 = M_{\rho,\omega}^2(i_{\rho,\omega}) = M_{\rho,\omega}^2(0) + i_{\rho,\omega}\,\sigma_{\rho,\omega}^2$$

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similarly for "excited pions", described by a Regge-like model:

$$m_\pi^2(n) = \begin{cases} m_{\pi^0}^2 & n=0, \\ m_0^2 + n\,\sigma_\pi^2 & n \geq 1, \end{cases}$$

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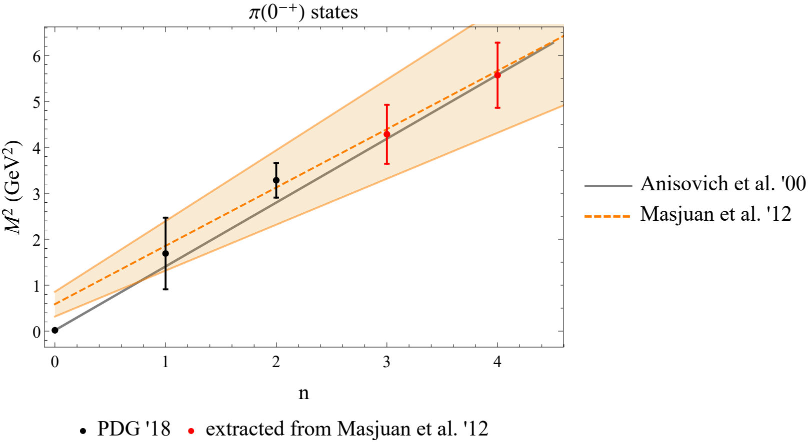
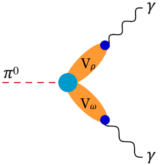


A Regge-like large-N_C inspired model



$$F_{\pi^{(n)}\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho,V_\omega} \frac{F_{V_\rho}(q_1^2)\,F_{V_\omega}(q_2^2)\,G_{\pi^{(n)}\,V_\rho\,V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

coupling between pions, and rho's and omega's taken diagonal for simplicity:

$$G_{\pi^{(n)}\,V_\rho\,V_\omega}(q_1^2,q_2^2) \propto \delta_{n\,i_\rho}\delta_{n\,i_\omega}$$

Satisfying short-distance constraints

$$\lim_{Q_3\to\infty}\lim_{\tilde{Q}\to\infty}\sum_{n=0}^{\infty}\frac{F_{\pi^{(n)}\gamma^*\gamma^*}(\tilde{Q}^2,\tilde{Q}^2)\,F_{\pi^{(n)}\gamma\gamma^*}(Q_3^2)}{Q_3^2+m_{\pi^{(n)}}^2}=$$
$$=\frac{1}{6\pi^2}\frac{1}{\tilde{Q}^2}\frac{1}{Q_3^2}+\mathcal{O}\left(\tilde{Q}^{-2}Q_3^{-4}\right),$$

where $F_{\pi^{(n)}\gamma^*\gamma^*}$ is the TFF of the n-th radially-excited pion

The infinite sum over excited pions changes the large-Q_3^2 behaviour from Q_3^{-4} (single pion pole) to Q_3^{-2}

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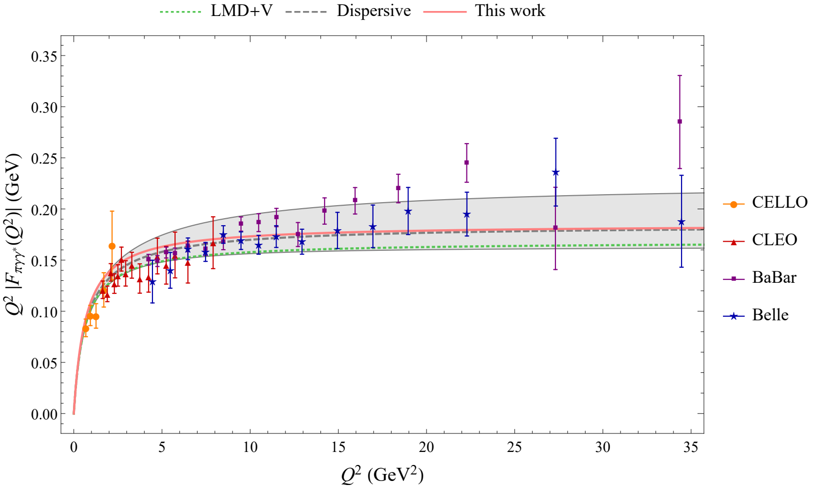
Is this a realistic model? Can it satisfy all theory constraints (anomaly, Brodsky-Lepage, etc.)?

Comparing our Regge-like model to phenomenology



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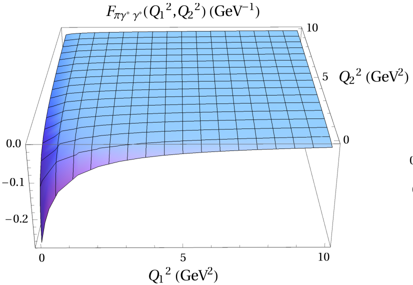
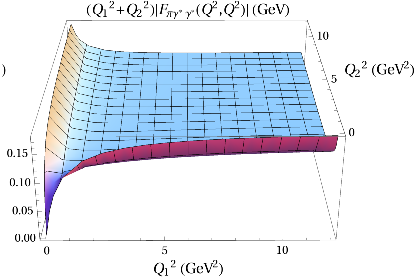
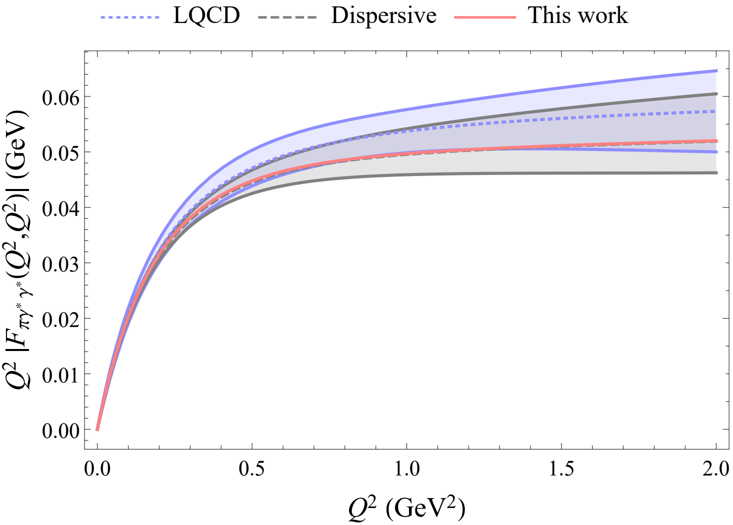




Comparing our model to the dispersive representation



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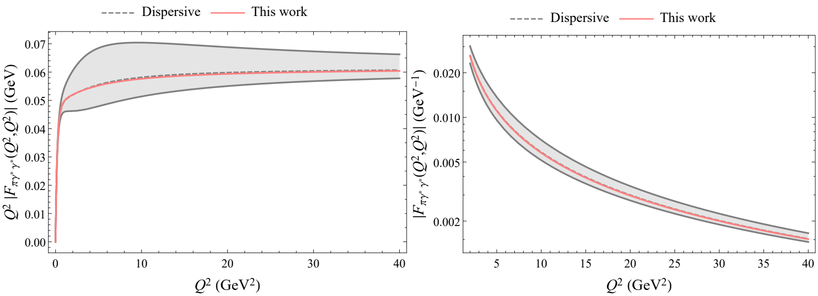

Contribution to $(g-2)_\mu$

The π^0-pole contribution to $(g-2)_\mu$ evaluated with our model is:

$$a_\mu^{\pi^0} = 64.1 \cdot 10^{-11}$$

very close to the value obtained with the dispersive representation for the pion TFF ($62.6^{+3.0}_{-2.5} \cdot 10^{-11}$)

After resumming the contribution of all pion excitations we get:

$$\Delta a_\mu^{\pi} := \sum_{n=1}^{\infty} a_\mu^{\pi^{(n)}} = 5.1(5) \cdot 10^{-11}$$

Much smaller than the shift obtained by Melnikov-Vainshtein by dropping the pion TFF at the outer $\pi^0\gamma^*\gamma$ vertex:

$$\Delta a_\mu^{\pi}(\text{M-V}) = 13.5 \cdot 10^{-11}$$

Contribution to $(g-2)_\mu$

The π^0-pole contribution to $(g-2)_\mu$ evaluated with a second model (not described here) is:

$$a_\mu^{\pi^0} = 64.5 \cdot 10^{-11}$$

very close to the value obtained with the dispersive representation for the pion TFF ($62.6^{+3.0}_{-2.5} \cdot 10^{-11}$)

After resumming the contribution of all pion excitations we get:

$$\Delta a_\mu^{\pi} := \sum_{n=1}^{\infty} a_\mu^{\pi^{(n)}} = 5.1(1) \cdot 10^{-11}$$

Much smaller than the shift obtained by Melnikov-Vainshtein by dropping the pion TFF at the outer $\pi^0\gamma^*\gamma$ vertex:

$$\Delta a_\mu^{\pi}(\text{M-V}) = 13.5 \cdot 10^{-11}$$

Effect due to short-distance constraints

Melnikov-Vainshtein's solution to satisfy (longitudinal) SDC: drop the π-TFF at the outer $\pi^0\gamma^*\gamma$ vertex. Effect is significant:

$$\Delta a_\mu^\pi(\text{M-V}) = 13.5 \cdot 10^{-11}$$

With two different models which satisfy the SDC, agree w/ data on the π^0 TFF and with the dispersive representation we obtain:

$$\Delta a_\mu^\pi(\text{our model}) \sim 5 \cdot 10^{-11}$$

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Work on the transverse SDC is in progress, but M-V estimate (axials) seems to be an overestimate (for various reasons)

Our models will be matched to the quark loop (in progress)

Improvements obtained with the dispersive approach

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
" " + subl. in N_c	–	–	–	0 ± 10	–	–	–
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Results with the dispersive approach:

Pion pole:	$62.6^{+3.0}_{-2.6}$	
Pion box:	-15.9 ± 0.2	
Kaon box (VMD):	~ -0.5	(prelim. Hoferichter, Stoffer)
Pion S-wave rescatt.:	-8 ± 1	
Longitudinal SDC (π^0):	~ 5	(prelim. $\eta^{(\prime)}$ in progr.)

Outline

Introduction

Setting up the stage: Master Formula

A dispersion relation for HLbL
- Pion-pole contribution
- Pion-box contribution
- Pion rescattering contribution

Short-distance constraints

Outlook and Conclusions

Conclusions

- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- the relevant measurable quantity entering the dispersion relation depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes

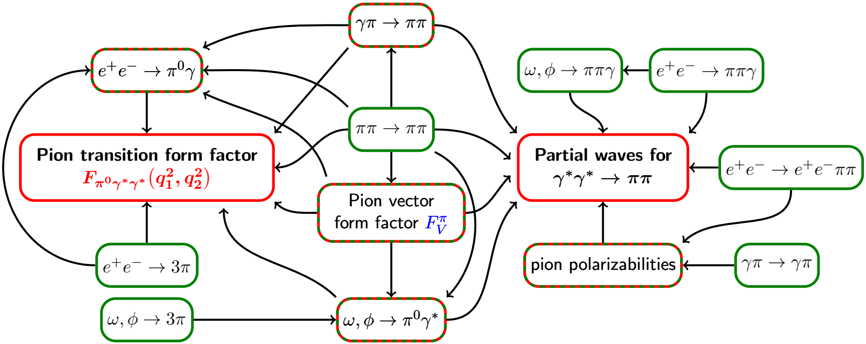
  these three contributions (S-wave for the latter) have been calculated with remarkably small uncertainties
- work on calculating other contributions and estimating missing pieces is in progress

Outlook

- More work is needed to complete the evaluation of contributions of 2π intermediate states esp. for $\ell \geq 2$
  - take into account experimental constraints on $\gamma^{(*)}\gamma \to \pi\pi$
  - estimate the dependence on the q^2 of the second photon (theoretically, there are no data on $\gamma^*\gamma^* \to \pi\pi$ – Lattice?)
  - $\Rightarrow$ solve the dispersion relation for the helicity amplitudes of $\gamma^*\gamma^* \to \pi\pi$, including a full treatment of the LHC
- same formulae apply to heavier $n \leq 2$ intermediate states ($\eta^{(\prime)}$ or $\bar{K}K$); for $n > 2$ the formalism must be extended;
- implementation of short-distance constraints is in progress: effect seems to be somewhat smaller than estimated so far

Hadronic light-by-light: a roadmap

GC, Hoferichter, Kubis, Procura, Stoffer arXiv:1408.2517 (PLB '14)



Artwork by M. Hoferichter

A reliable evaluation of the HLbL requires many different contributions by and a collaboration among (lattice) theorists and experimentalists