Beam-Beam Compensation in a Collider Based on Energy Recovery Linac and Storage Ring

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Abstract. One of the aims of new circular collider projects is further increase in their luminosity. The high space charge electromagnetic field at the meeting points limits the achievable current densities and consequently the luminosity. Non-linear focusing compensation in a storage ring done by the opposite-charge beam circulating in another storage ring was proposed and tested many years ago. Ya. S. Derbenev has shown that such a scheme suffers from tune shifts of coherent betatron oscillations, which move betatron frequencies toward the nearest integer or half-integer resonance. In this paper, the collider based on electron energy recovery linac (ERL) and "figure-8" positron storage ring with beams of equal currents is considered. Positrons are circulating in a two-loop storage ring (positron-positron collider), and electron-electron collider uses ERL, as in original Tigner's proposal. Thus, a collision of four bunches and space-charge compensation in a multi-bunch mode can be ensured. The mathematical and numerical analysis of this configuration is presented.

INTRODUCTION

Compensating for non-linear focusing in the storage ring colliders due to an opposite-charge beam circulating in another storage ring was proposed and tested many years ago (see [1] and references there). Ya.S. Derbenev had shown first [2] that the scheme suffers from tune shifts of coherent betatron oscillations, which move betatron frequencies toward the nearest integer or half-integer resonance. Beam-beam compensation in an installation with two "figure-8" storage rings (see Fig. 1) was considered in [3].

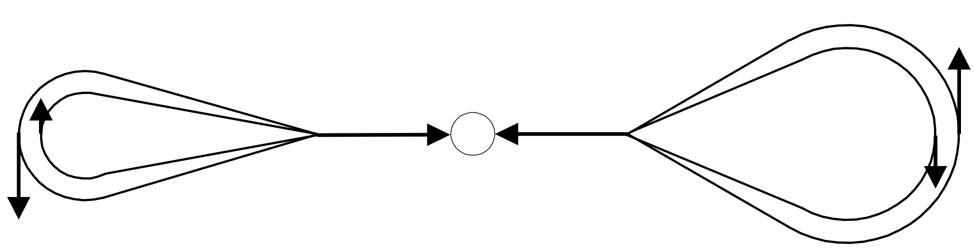

FIGURE 1. Scheme of two-beam compensated electron-positron collider

There are two "figure-8" storage rings of different energies, the electron one and the positron one. The collision takes place twice per turn in the common straight section with the meeting point. The case of equal beam energies can be realized in one "figure-8" storage ring with proper electrostatic separation. Certainly, other combinations of particles with opposite electric charges are possible. The straight-forward modification of this scheme is replacement of electron storage ring by an ERL (see Fig. 2).

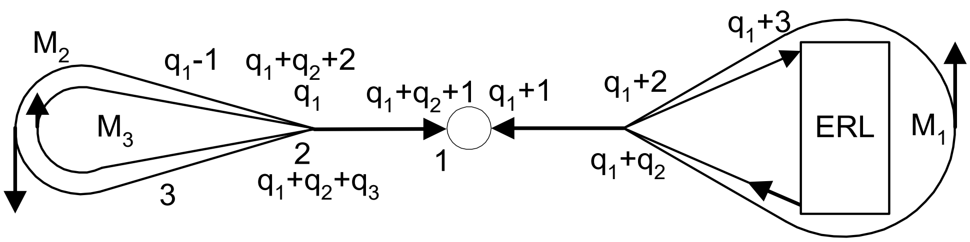

FIGURE 2. Positron storage ring and ERL

In this paper we analyse the stability condition of this beam-beam compensation scheme.

CHARACTERISTIC EQUATION

As in [3], we will consider the case of periodic bunches (the period T may be an integer of the radiofrequency period).), q_1 bunches in the left (from the meeting point) loop of the storage ring, q_2 in the right loop, and q_3 in the left ERL loop.). It means that the times of flight through the left and the right positron (storage ring) loops, and the left electron (ERL) loop are q_1T, q_2T and q_3T, respectively. To simplify equations, we assume bunches to have the same charge. Since energies of the ring and ERL are different, the description of the particle states will be done with the help of the dimensionless momentum $\beta\gamma x'$, not the angle x', taken as the second variable in the column. In the model of rigid bunches (see, e. g., [4]), the beams state is represented by a column of $2(q_1 + q_2 + q_3)$ components. Then the $2(q_1 + q_2 + q_3) \times 2(q_1 + q_2 + q_3)$ square matrix **TMS** [2, 3] describes the transformation in one period, and is the product of the following three matrices. The matrix **S**, which describes collision of four bunches, is

$$\mathbf{S} = \begin{pmatrix} E & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & E & 0 & 0 & 0 & -C & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E \\ -C & 0 & 0 & 0 & 0 & 0 & 0 & C & 0 & 0 \end{pmatrix} \qquad (1)$$

where E is the 2×2 unity matrix, 0 is 2×2 zero matrix,

$$C = \begin{pmatrix} 0 & 0 \\ -d & 0 \end{pmatrix} \qquad (2)$$

and d is the interaction parameter (the optical strength of the beam field focusing multiplied by $\beta\gamma$). To clarify notations, we have chosen $q_1 = 3$, $q_2 = 4$, $q_3 = 3$ in Eqs. (1, 3-5).

$$\mathbf{T} = \begin{pmatrix} 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E \end{pmatrix} \quad (3)$$

changes the bunch longitudinal coordinate in one period, and

$$\mathbf{M} = \begin{pmatrix} M_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_3 \end{pmatrix} \quad (4)$$

contains transverse matrices of right positron loop M_1, left positron loop M_2 and electron loop M_3. To find the stability condition, one has to calculate eigenvalues of matrix **TMS**. The corresponding characteristic equation is

$$0 = \left|TMS - \lambda E\right| = \left|MS - \lambda T^{-1}\right| =$$
$$\begin{vmatrix} M_1 & 0 & 0 & M_1C & 0 & 0 & -\lambda E & 0 & 0 & 0 \\ -\lambda E & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda E & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_2C & 0 & -\lambda E & M_2 & 0 & 0 & 0 & -M_2C & 0 & 0 \\ 0 & 0 & 0 & -\lambda E & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda E & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda E & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda E & E & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda E & E \\ -M_3C & 0 & 0 & 0 & 0 & 0 & 0 & M_3C & 0 & -\lambda E \end{vmatrix} \quad (5)$$

Simplification of this $2(q_1 + q_2 + q_3) \times 2(q_1 + q_2 + q_3)$ determinant leads to 6×6 determinant

$$0 = \left|MS - \lambda T^{-1}\right| = \begin{vmatrix} M_1 & M_1 C - \lambda^{q_2} E & 0 \\ M_2 C - \lambda^{q_1} E & M_2 & -M_2 C \\ -M_3 C & 0 & M_3 C - \lambda^{q_3} E \end{vmatrix} \tag{6}$$

It can be represented as the second-order polynomial of the interaction parameter d

$$\begin{gathered} 0 = 2f(\lambda) = \\ \lambda^{q_1+q_2} + \lambda^{-q_1-q_2} - \mathrm{Sp}(M_1 M_2) + d\left[(M_1)_{12}\left(\lambda^{q_1} + \lambda^{-q_1}\right) + (M_2)_{12}\left(\lambda^{q_2} + \lambda^{-q_2}\right)\right] + d^2 (M_1)_{12}(M_2)_{12} + \\ d(M_3)_{12}\lambda^{-q_3}\left[\lambda^{q_1+q_2} + \lambda^{-q_1-q_2} - \mathrm{Sp}(M_1 M_2)\right] + d^2 (M_3)_{12}\lambda^{-q_3}\left[\lambda^{q_1}(M_1)_{12} + (M_2)_{12}\lambda^{-q_2}\right] \end{gathered} \tag{7}$$

STABILITY CRITERION

The stability criterion can be expressed as (see, e. g., [5, 6]): "Mapping $w = f(\lambda)$ of the outer part $|\lambda| > 1$ of the unit disk does not contain the origin $w = 0$". The boundary of this area is

$$\begin{gathered} w(\varphi) = f\left(e^{i\varphi}\right) = \\ \cos\left[(q_1+q_2)\varphi\right] - \frac{\mathrm{Sp}(M_1 M_2)}{2} + d\left[(M_1)_{12}\cos(q_1\varphi) + (M_2)_{12}\cos(q_2\varphi)\right] + \frac{d^2 (M_1)_{12}(M_2)_{12}}{2} + \\ d(M_3)_{12} e^{-iq_3\varphi}\left\{\cos\left[(q_1+q_2)\varphi\right] - \frac{\mathrm{Sp}(M_1 M_2)}{2}\right\} + d^2 (M_3)_{12} e^{-iq_3\varphi}\frac{e^{iq_1\varphi}(M_1)_{12} + (M_2)_{12} e^{-iq_2\varphi}}{2} \end{gathered} \tag{8}$$

Expressing matrix elements through beta function β in the meeting point and betatron phase advances μ_2 and μ_2 one can rewrite Eq. (8) in the form

$$\begin{gathered} w(\varphi) = f\left(e^{i\varphi}\right) = \\ \cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2) + d\beta\left[\sin\mu_1 \cos(q_1\varphi) + \sin\mu_2\cos(q_2\varphi)\right] + \frac{(d\beta)^2 \sin\mu_1 \sin\mu_2}{2} + \\ d\beta\frac{(M_3)_{12}}{\beta} e^{-iq_3\varphi}\left\{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)\right\} + (d\beta)^2 \frac{(M_3)_{12}}{\beta} e^{-iq_3\varphi}\frac{\sin\mu_1 e^{iq_1\varphi} + \sin\mu_2 e^{-iq_2\varphi}}{2} \end{gathered} \tag{9}$$

The characteristic equation $w(\varphi) = 0$ at $0 \le \varphi < 2\pi$ describes two-dimensional surface in the four-dimensional space of the system parameters $\frac{(M_3)_{12}}{\beta}, d\beta, \mu_1, \mu_2$ at some fixed discrete parameters q_1, q_2, q_3. This surface divides the parameter domains with different number of eigenvalues λ with $|\lambda| > 1$. One of these domains, where there are no such big eigenvalues, is just the stability domain. The stability domain contains region of small interaction parameters $d\beta$ and $(M_3)_{12}$, which may be considered as proportional to the ERL beam current.

Therefore, one can investigate the section of stability domain by the upper half-plane of parameters $\frac{(M_3)_{12}}{\beta}, d\beta$ at some constant phase advances μ_1 and μ_2. Real and imaginary parts of characteristic equation Eq. (9) are

$$\begin{aligned}
0 &= 2\cos\left[(q_1+q_2)\varphi\right] - 2\cos(\mu_1+\mu_2) + \\
&2d\beta\left[\sin\mu_1\cos(q_1\varphi) + \sin\mu_2\cos(q_2\varphi)\right] + (d\beta)^2\sin\mu_1\sin\mu_2 + \\
&2d\beta\frac{(M_3)_{12}}{\beta}\cos(q_3\varphi)\left\{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)\right\} + \\
&(d\beta)^2\frac{(M_3)_{12}}{\beta}\sin\mu_1\cos\left[(q_1-q_3)\varphi\right] + (d\beta)^2\frac{(M_3)_{12}}{\beta}\sin\mu_2\cos\left[(q_2+q_3)\varphi\right]
\end{aligned} \tag{10}$$

$$\begin{aligned}
0 &= -2\frac{(M_3)_{12}}{\beta}\sin(q_3\varphi)\left\{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)\right\} + d\beta\frac{(M_3)_{12}}{\beta}\sin\mu_1\sin\left[(q_1-q_3)\varphi\right] - \\
&d\beta\frac{(M_3)_{12}}{\beta}\sin\mu_2\sin\left[(q_2+q_3)\varphi\right]
\end{aligned} \tag{11}$$

Equations (10) and (11) have the following solutions (separating curves on the plane $\frac{(M_3)_{12}}{\beta}, d\beta$):

1. at $\varphi \neq 0, \pi$ and $(M_3)_{12} \neq 0$

$$\frac{(M_3)_{12}}{\beta} = \frac{\sin^2(\mu_1+\mu_2)\dfrac{\sin(q_3\varphi)}{\sin\mu_1\sin(q_1\varphi) - \sin\mu_2\sin(q_2\varphi)} - \dfrac{\sin\mu_1\sin(q_1\varphi) - \sin\mu_2\sin(q_2\varphi)}{\sin(q_3\varphi)}}{2\left\{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)\right\}} \tag{12}$$

$$d\beta = \frac{2\sin(q_3\varphi)\left\{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)\right\}}{\sin\mu_1\sin\left[(q_1-q_3)\varphi\right] - \sin\mu_2\sin\left[(q_2+q_3)\varphi\right]} \tag{13}$$

2. at $(M_3)_{12} = 0$

$$\begin{aligned}
0 &= (d\beta)^2\sin\mu_1\sin\mu_2 + 2d\beta\left[\sin\mu_1\cos(q_1\varphi) + \sin\mu_2\cos(q_2\varphi)\right] + \\
&2\cos\left[(q_1+q_2)\varphi\right] - 2\cos(\mu_1+\mu_2)
\end{aligned} \tag{14}$$

or

$$\begin{aligned}
d\beta &= -\frac{\cos(q_1\varphi)}{\sin\mu_2} - \frac{\cos(q_2\varphi)}{\sin\mu_1} \pm \sqrt{\left[\frac{\cos(q_1\varphi)}{\sin\mu_2} + \frac{\cos(q_2\varphi)}{\sin\mu_1}\right]^2 - 2\frac{\cos\left[(q_1+q_2)\varphi\right] - \cos(\mu_1+\mu_2)}{\sin\mu_1\sin\mu_2}} = \\
&-\frac{\sin\mu_1\cos(q_1\varphi) + \sin\mu_2\cos(q_2\varphi) \pm \sqrt{\sin^2(\mu_1+\mu_2) - \left[\sin\mu_1\sin(q_1\varphi) - \sin\mu_2\sin(q_2\varphi)\right]^2}}{\sin\mu_1\sin\mu_2}
\end{aligned} \tag{15}$$

3. at $\varphi = 0$

$$\frac{(M_3)_{12}}{\beta} = -\frac{1}{d\beta} - \frac{\sin\mu_1 + \sin\mu_2 + d\beta\sin\mu_1\sin\mu_2}{2 - 2\cos(\mu_1 + \mu_2) + d\beta(\sin\mu_1 + \sin\mu_2)} \tag{16}$$

4. at $\varphi = \pi$

$$-(-1)^{q_3}\frac{(M_3)_{12}}{\beta} = \frac{1}{d\beta} + \frac{\sin\mu_1(-1)^{q_1} + \sin\mu_2(-1)^{q_2} + d\beta\sin\mu_1\sin\mu_2}{2(-1)^{q_1+q_2} - 2\cos(\mu_1 + \mu_2) + d\beta\left[\sin\mu_1(-1)^{q_1} + \sin\mu_2(-1)^{q_2}\right]} \tag{17}$$

The curve described by Eq. (10), Eq. (11) and the straight line $(M_3)_{12} = 0$ divide the half-plane of parameters $\frac{(M_3)_{12}}{\beta}, d\beta$. To define the stability region let us consider the solution of characteristic equation near the origin, i.e. at small $\frac{|(M_3)_{12}|}{\beta}$ and $d\beta$.

Let $\lambda = e^{i\varphi}$, and $\lambda_0 = e^{i\varphi_0}$ is solution of Eq. (14). Then expansion of the characteristic equation Eq. (7) over $d(M_3)_{12}$

$$(q_1 + q_2)(\varphi - \varphi_0)\sin\left[(q_1 + q_2)\varphi_0\right] \approx -\beta^2 d^2 \frac{(M_3)_{12}}{\beta} e^{-iq_3\varphi_0} \frac{e^{-iq_1\varphi_0}\sin\mu_1 + \sin\mu_2 e^{iq_2\varphi_0}}{2} \tag{18}$$

gives the stability criterion (positive damping rate of betatron oscillations)

$$\begin{aligned} &0 \le (q_1 + q_2)\,\mathrm{Im}(\varphi - \varphi_0) \approx \\ &\beta^2 d^2 \frac{(M_3)_{12}}{\beta} \frac{\sin\mu_1 \sin\left[(q_1 + q_3)\varphi_0\right] + \sin\mu_2 \sin\left[(q_3 - q_2)\varphi_0\right]}{2\sin\left[(q_1 + q_2)\varphi_0\right]} \approx, \\ &\beta^2 d^2 \frac{(M_3)_{12}}{\beta} \frac{\sin\mu_1 \sin\left(\frac{q_1 + q_3}{q_1 + q_2}\mu\right) - \sin\mu_2 \sin\left(\frac{q_2 - q_3}{q_1 + q_2}\mu\right)}{2\sin\mu} \end{aligned} \tag{19}$$

or

$$0 \le (M_3)_{12} \frac{\sin\mu_1 \sin\left(\frac{q_1 + q_3}{q_1 + q_2}\mu\right) - \sin\mu_2 \sin\left(\frac{q_2 - q_3}{q_1 + q_2}\mu\right)}{\sin\mu}, \tag{20}$$

where $\mu = (q_1 + q_2)\varphi_0 \approx \mu_1 + \mu_2$ is the betatron phase advance. The sign of the fraction in Eq. (20) defines the stable quadrant ($(M_3)_{12} > 0$ or $(M_3)_{12} < 0$) of the parameter half-plane.

The corresponding betatron tune shift is

$$\Delta\nu_{ERL} = \frac{(q_1+q_2)\operatorname{Re}(\varphi-\varphi_0)}{2\pi} \approx -d^2 (M_3)_{12} \frac{(M_1)_{12}\cos\left(\frac{q_1+q_3}{q_1+q_2}\mu\right)+(M_2)_{12}\cos\left(\frac{q_2-q_3}{q_1+q_2}\mu\right)}{4\pi\sin\mu} \tag{21}$$

The stability region for $q_1 = q_2 = q_3 = 1$ $\mu_1 = 0.05\cdot 2\pi$ and $\mu_2 = 0.15\cdot 2\pi$ is shown in Fig. 3.

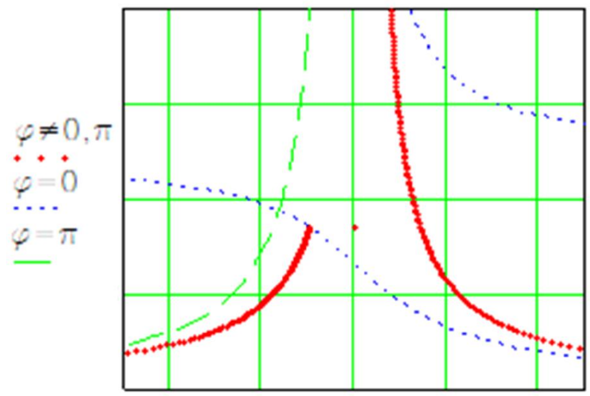

FIGURE 3. Plot $d\beta\left[\frac{(M_3)_{12}}{\beta}\right]$. The stability region lays in left quadrant at low $d\beta$.

The stability region is the curvilinear quadrangle in the left quadrant limited by the coordinate axes and curves Eq. (12) and Eqs. (16, 17). The maximum stable $d\beta$ value

$$(d\beta)_{\max} = 2q_3 \frac{(-1)^{q_1+q_2} - \cos(\mu_1+\mu_2)}{\sin\mu_1 (-1)^{q_1}(q_1-q_3) - \sin\mu_2 (-1)^{q_2}(q_2+q_3)} \tag{22}$$

takes place at $\varphi = \pi$ and

$$\frac{(M_3)_{12}}{\beta} = (-1)^{q_3} \frac{\sin^2(\mu_1+\mu_2)\dfrac{q_3}{q_1(-1)^{q_1}\sin\mu_1 - q_2(-1)^{q_2}\sin\mu_2} - \dfrac{q_1(-1)^{q_1}\sin\mu_1 - q_2(-1)^{q_2}\sin\mu_2}{q_3}}{2\left\{(-1)^{q_1+q_2} - \cos(\mu_1+\mu_2)\right\}} \tag{23}$$

The surface plot of $d\beta_{\max}$ at $0.05\cdot\pi < \mu_1 < 0.95\cdot\pi$ and $0.05\cdot\pi < \mu_2 < 0.95\cdot\pi$ is shown in Fig. 4.

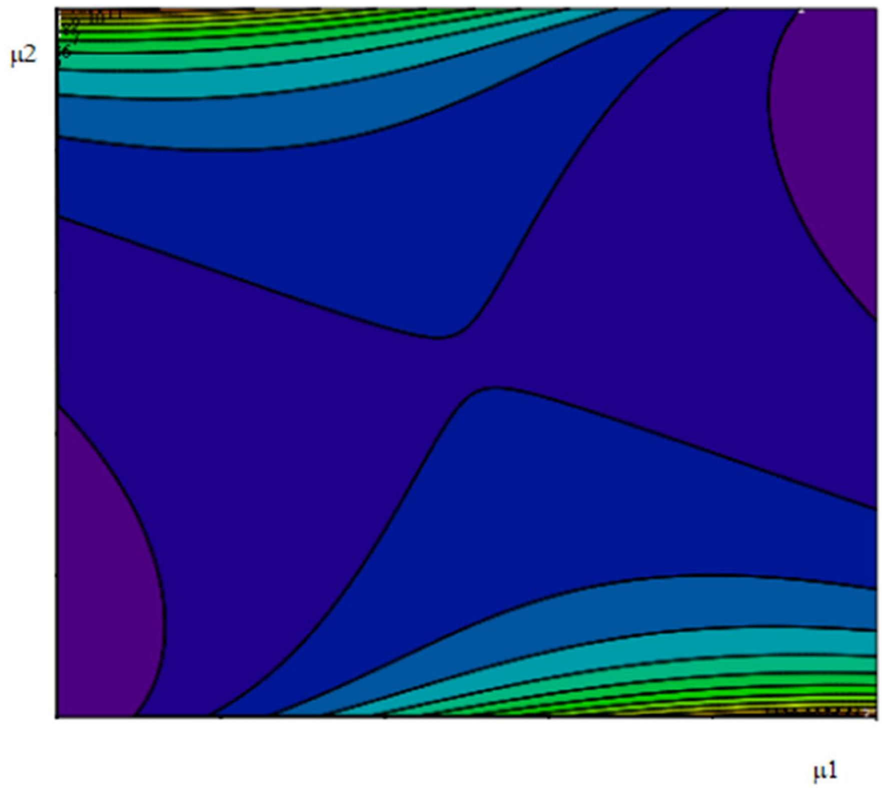

FIGURE 4. The surface plot of $(d\beta)_{\max}$ at $0.05\cdot\pi < \mu_1 < 0.95\cdot\pi$ (abscises) and $0.05\cdot\pi < \mu_2 < 0.95\cdot\pi$ (ordinates)

It is worth noting that the interesting, according to Eq. (22), case $q_1 = q_3$ and $\mu_2 = 0$ is stable at low currents. Therefore, the stability may take place at high $(d\beta)_{\max}$, i. e., currents, also.

CONCLUSION

In this paper we derived the stability criterion of beam-beam compensation scheme with electron ERL and positron "figure 8" storage ring. Preliminary estimates show the possibility to achieve the stability at high enough beam currents.

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