Mass spectra in $\mathcal{N} = 1$ SQCD with additional colorless but flavored fields

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Abstract

Considered is the direct $\mathcal{N} = 1$ SQCD (i.e. supersymmetric QCD) - like Φ-theory with $SU(N_c)$ colors and $3N_c/2 < N_F < 2N_c$ flavors of light quarks $\overline{Q}_j^{\,b}$, Q_a^i, $a, b = 1...N_c$, $i, j = 1...N_F$, with small mass parameter $0 < m_Q \ll \Lambda_Q$. Besides, it includes N_F^2 additional colorless but flavored fields Φ_i^j, with the large mass parameter $\mu_\Phi \gg \Lambda_Q$, interacting with quarks through the Yukawa coupling in the superpotential. In parallel, is considered its Seiberg's dual variant, i.e. the $d\Phi$-theory with $\overline{N}_c = (N_F - N_c)$ dual colors and $3N_c/2 < N_F < 2N_c$ flavors of dual quarks $\overline{q}_b^{\,j}$, q_i^a. The multiplicities of various vacua and values of the quark and gluino condensates in all vacua are found.

It is shown that in considered vacua of both the direct and dual theories the quarks are in the conformal regimes at scales $\mu < \Lambda_Q$. The dynamics of these regimes is sufficiently simple and well understood, so that no additional dynamical assumptions were needed to calculate the mass spectra. It is shown that **mass spectra of the direct Φ and dual $d\Phi$ - theories are different**, in disagreement with the Seiberg hypothesis about complete equivalence of such two theories.

Besides it is shown in the direct Φ-theory that a qualitatively new phenomenon takes place: the seemingly heavy and dynamically irrelevant fields Φ '**return back**' and there appear **two additional generations of light Φ-particles** with small masses $\mu_{2,3}^{\rm pole}(\Phi) \ll \Lambda_Q$.

Contents

1 Definitions and some generalities

Direct Φ - theory

The field content of this direct $\mathcal{N}=1$ Φ - theory includes $SU(N_c)$ gluons and $3N_c/2 < N_F < 2N_c$ flavors of light quarks $\overline{Q}_j, Q^i$. Besides, there are N_F^2 colorless but flavored fields Φ_i^j (fions) with the large mass parameter $\mu_\Phi \gg \Lambda_Q$.

The Lagrangian at the scale $\mu=\Lambda_Q$ in superfield notations looks as (the exponents with gluons in the Kahler term K are implied here and everywhere below, $\overline{N}_c = N_F - N_c$):

$$L = \int d^4x \int d^2\overline{\theta} d^2\theta\, K(x,\overline{\theta},\theta) + \left(\mathcal{W} = \int d^4x \int d^2\theta\, \mathcal{W}_{tot}(x,\theta) + h.c.\right), \tag{1.1}$$

$$K = \mathrm{Tr}\left(\Phi^\dagger\Phi\right) + \mathrm{Tr}\Big(Q^\dagger Q + (Q\to\overline{Q})\Big), \quad \mathcal{W}_{tot} = \mathcal{W}_{\rm gauge} + \mathcal{W}_{\rm matter}\,, \quad \mathcal{W}_{gauge} = \frac{2\pi}{\alpha(\mu,\Lambda_Q)} S\,,$$

$$\mathcal{W}_{\rm matter} = \mathcal{W}_Q + \mathcal{W}_\Phi\,, \quad \mathcal{W}_Q = \mathrm{Tr}\,\overline{Q} m_Q^{\rm tot} Q = \mathrm{Tr}\,\overline{Q}(m_Q - \Phi)Q, \quad \mathcal{W}_\Phi = \frac{\mu_\Phi}{2}\left[\mathrm{Tr}\,(\Phi^2) - \frac{1}{\overline{N}_c}\Big(\mathrm{Tr}\,\Phi\Big)^2\right].$$

Here: μ_Φ and m_Q are the mass parameters, $S = W^a_\beta W^{a,\beta}/32\pi^2$ where W^a_β is the field strength of the gauge superfield, $a = 1...N_c^2-1$, $\beta = 1,2$, $\alpha(\mu,\Lambda_Q) = g^2(\mu,\Lambda_Q)/4\pi$ is the gauge coupling with its scale factor Λ_Q. This normalization of fields is used everywhere below in the text.

In the usual notations the Lagrangian $L = T - V$ looks as

$$T_{boson} = \int d^4x \left[-\frac{1}{4g^2} \mathrm{Tr}(G^{\,a}_{\mu\nu})^2 + \mathrm{Tr}\Big((D_\mu Q)^\dagger (D_\mu Q) + (Q \to \overline{Q}) + (\partial_\mu \Phi)^\dagger \partial_\mu \Phi \Big) \right],$$

$$V_{boson} = \int d^4x \left[\mathrm{Tr}\left(g^2 \Big(Q^\dagger T^a Q - \overline{Q}^\dagger T^a \overline{Q} \Big)^2 + \Big| (m_Q - \Phi) Q \Big|^2 + \Big| \overline{Q} (m_Q - \Phi) \Big|^2 + \left| \frac{\partial \mathcal{W}_\Phi}{\partial \Phi^j_i} - \overline{Q}_j Q^i \right|^2 \right) \right],$$

$$L_{fermion} = \int d^4x \left[\mathrm{Tr}\left(\frac{1}{g^2} \overline{\lambda} i \sigma_\mu D_\mu \lambda + \overline{\chi}\, i \sigma_\mu D_\mu \chi + \overline{\tilde{\chi}}\, i \sigma_\mu D_\mu \tilde{\chi} \right) + \mathrm{Tr}\left(\tilde{\chi} (m_Q - \Phi) \chi + (h.c.) \right) + \right.$$

$$+ \mathrm{Tr}\left((Q^\dagger \lambda^a T^a \chi + h.c.) - (Q \to \overline{Q},\, \chi \to \tilde{\chi}) \right) +$$

$$\left. + \mathrm{Tr}\left(\overline{Q} \psi \chi + \tilde{\chi} \psi Q + (h.c.) \right) \right], \qquad a = 1 ... N_c^2 - 1\,, \quad i, j = 1 ... N_F\,.$$

Dual dΦ - theory

In parallel with the direct Φ - theory with $3N_c/2 < N_F < 2N_c\,,$ we consider also the Seiberg dual variant (the $d\Phi$ - theory). The dual Lagrangian at $\mu=\Lambda_Q$ looks as

$$\overline{K}=\mathrm{Tr}\,\Phi^\dagger\Phi+\mathrm{Tr}\Bigl(q^\dagger q+(q\to\overline{q})\Bigr)+\mathrm{Tr}\,\frac{M^\dagger M}{f^2 Z_q^2\Lambda_Q^2}\,,\quad \overline{\mathcal{W}}=\overline{\mathcal{W}}_{\rm gauge}+\overline{\mathcal{W}}_{\rm matter},$$

$$\overline{\mathcal{W}}_{\rm gauge}=-\,\frac{2\pi}{\overline{\alpha}(\mu=\Lambda_Q)}\,\overline{S},\quad \overline{\mathcal{W}}_{\rm matter}=\mathcal{W}_\Phi+\overline{\mathcal{W}}_M+\mathcal{W}_q\,, \tag{1.2}$$

$$\overline{\mathcal{W}}_M=\mathrm{Tr}\,M(m_Q-\Phi),\quad \mathcal{W}_q=-\,\frac{1}{Z_q\Lambda_Q}\,\mathrm{Tr}\Bigl(\overline{\rm q}\,{\rm M}\,{\rm q}\Bigr)\,.$$

Here: the number of dual colors is $\overline{N}_c=N_F-N_c$, $\overline{\rm b}_{\rm o}=3\overline{N}_c-N_F$, and $M^i_j\to(\overline{Q}_j Q^i)$ are the N_F^2 elementary mion fields, $\overline{a}(\mu)=\overline{N}_c\overline{g}^2(\mu)/8\pi$ is the dual running gauge coupling (with its scale parameter $|\Lambda_q|=\Lambda_Q$), $\overline{S}=\overline{\rm W}^{\,b}_\beta\overline{\rm W}^{\,b,\,\beta}/32\pi^2$, $\overline{\rm W}^{\,b}_\beta$ is the dual gluon field strength. The factors $a_f=\overline{N}_c f^2/8\pi^2$ and Z_q in (1.2) are $O(1)$ at $\overline{\rm b}_{\rm o}/N_F=O(1)$, but are parametrically small at $\overline{\rm b}_{\rm o}/N_F\ll 1$ (see Conclusions).

At $3/2 < N_F/N_c < 2$ this dual theory can be taken as UV free at $\mu \gg \Lambda_Q$. We consider below this dual theory at $\mu \leq \Lambda_Q$ only where, according to Seiberg's hypothesis, it is equivalent to the direct Φ - theory.

Really, **the fields Φ remain always too heavy and dynamically irrelevant in this $d\Phi$ - theory** at $3N_c/2 < N_F < 2N_c$ and $\mu < \Lambda_Q$, so that they can be integrated out once and forever and, finally, we write the Lagrangian of the dual theory at $\mu = \Lambda_Q$ in the form

$$K = \mathrm{Tr}\Big(q^\dagger q + (q \to \overline{q})\Big) + \mathrm{Tr}\,\frac{M^\dagger M}{f^2 Z_q^2 \Lambda_Q^2}\,, \quad \overline{\mathcal{W}}_{\rm matter} = \mathcal{W}_M + \mathcal{W}_q\,,$$

$$\mathcal{W}_M = m_Q \mathrm{Tr}\, M - \frac{1}{2\mu_\Phi}\Bigg[\mathrm{Tr}\,(M^2) - \frac{1}{N_c}(\mathrm{Tr}\, M)^2\Bigg]\,, \quad \mathcal{W}_q = -\,\frac{1}{Z_q \Lambda_Q}\,\mathrm{Tr}\Big(\overline{\mathrm{q}}\,\mathrm{M}\,\mathrm{q}\Big)\,. \tag{1.3}$$

The gluino condensates of the direct and dual theories are matched in all vacua, $\langle -\overline{S}\rangle = \langle S\rangle = \Lambda^3_{YM}$, as well as $\langle M^i_j(\mu = \Lambda_Q)\rangle = \langle M^i_j\rangle = \langle \overline{Q}_j Q_i(\mu = \Lambda_Q)\rangle = \langle \overline{Q}_j Q_i\rangle$.

Besides, the perturbative NSVZ β-function for (effectively) massless SUSY theories is used

$$\frac{d}{d\ln\mu}\,\frac{1}{a(\mu)}=\beta(a)=\frac{1}{1-a(\mu)}\Big[\frac{\mathrm{b_o}}{N_c}-\frac{N_F}{N_c}\gamma_Q(a)\Big]\,,\quad a(\mu)=\frac{N_c g^2}{8\pi^2}\,,\quad \mathrm{b_o}=3N_c-N_F\,, \tag{1.4}$$

where γ_Q is the quark anomalous dimension (and similarly in the dual theory: $\gamma_Q\to\gamma_q$, $a\to\overline{a}=\overline{N}_c\overline{g}^2/8\pi^2$, $a_f=\overline{N}_c f^2/8\pi^2$, $\mathrm{b_o}\to\overline{\mathrm{b}}_\mathrm{o}=(3\overline{N}_c-N_F)$).

We take below (except for Conclusions): $\mathrm{b_o}/N_F$ and $\overline{\mathrm{b}}_\mathrm{o}/N_F$ as $O(1)$. Then Z_q and a_f are both $O(1)$ and are omitted. In both the direct and dual theories with $3N_c/2<N_F<2N_c$ **the regime is conformal** with frozen couplings at $\mu<\Lambda_Q$: $a(\mu<\Lambda_Q)=a_*=O(1)$, $\overline{a}(\mu<\Lambda_Q)=\overline{a}_*=O(1)$, $a_f(\mu<\Lambda_Q)=a_f^*=O(1)$ (until it is broken by particles masses at lower energies). Then, the anomalous dimensions of all fields and so the corresponding renormalization factors of all Kahler terms are known in the conformal regime

$$\beta_{\rm conf}(a_*)=0\;\to\;\gamma_Q(a_*)=\frac{3N_c-N_F}{N_F},\;\;\gamma_\Phi(a_*)=-2\gamma_Q(a_*),\;\;\gamma_q(\overline{a}_*)=\frac{3\overline{N}_c-N_F}{N_F},\;\;\gamma_M(\overline{a}_*)=-2\gamma_q(\overline{a}_*), \tag{1.5}$$

in the direct and dual theories respectively.

2 Quark and gluino condensates and multiplicities of vacua at $3N_c/2 < N_F < 2N_c$

To obtain the numerical values of the quark condensates (really, the mean vacuum values) $\langle \overline{Q}_j Q^i \rangle = \delta^i_j \langle (\overline{Q}Q)_i \rangle$ (**but only for this purpose**), the simplest way is to use the known **exact form** of the non-perturbative contribution $\mathcal{W}_{\rm non-pert}$ to the effective superpotential in the standard SQCD (i.e. without the fion fields Φ). It seems clear that at sufficiently large values of $\mu_\Phi \gg \Lambda_Q$ among the vacua of the Φ-theory there will be N_c vacua of the standard SQCD in which, definitely, all fions Φ are too heavy and dynamically irrelevant. Therefore, they all can be integrated out and this only results in additional 4-quark term in the superpotential, so that **the exact** effective superpotential will look as

$$\mathcal{W}_{\rm eff} = \Bigg[\mathcal{W}_{\rm non-pert} = -\overline{N}_c S = -\overline{N}_c\Big(\frac{\det \overline{Q}Q}{\Lambda_Q^{\rm b_o}}\Big)^{1/\overline{N}_c}\Bigg] + m_Q {\rm Tr}\,\overline{Q}Q - \frac{1}{2\mu_\Phi}\Bigg[{\rm Tr}(\overline{Q}Q)^2 - \frac{1}{N_c}({\rm Tr}\,\overline{Q}Q)^2\Bigg], \qquad (2.1)$$

where the first non-perturbative term in (2.1) is well known in the standard $\mathcal{N}=1$ SQCD without fions.

Indeed, e.g. at $3N_c/2 < N_F < 2N_c$ and sufficiently large μ_Φ, there are N_c SQCD vacua in (2.1) with the unbroken $U(N_F)$ global flavor symmetry. In these, the last 4-quark term in (2.1) gives a small correction only and can be neglected and one obtains the well known results

$$\langle \overline{Q}_j Q^i \rangle_{QCD} \approx \delta^i_j \frac{1}{m_Q}\Big(\Lambda_{YM}^{(\rm SQCD)}\Big)^3 = \delta^i_j \frac{1}{m_Q}\Big(\Lambda_Q^{\rm b_o} m_Q^{N_F}\Big)^{1/N_c}, \quad \langle S \rangle_{QCD} = \langle \frac{\lambda\lambda}{32\pi^2} \rangle_{QCD} \approx \Big(\Lambda_Q^{\rm b_o} m_Q^{N_F}\Big)^{1/N_c}. \qquad (2.2)$$

Now, using the holomorphic dependence of the superpotential (2.1) on the chiral superfields $(\overline{Q}_j Q^i)$ and the chiral parameters m_Q and μ_Φ, the exact form (2.1) can be used to find the values of the quark condensates $\langle\overline{Q}_j Q^i\rangle$ in all other numerous vacua of the Φ - theory and at all other values of $\mu_\Phi > \Lambda_Q$. It is worth recalling only that, in general, as in the standard SQCD, $\mathcal{W}_{\rm eff}$ **is not the superpotential of the genuine low energy Lagrangian describing lightest particles, it determines only the values of the vacuum condensates** $\langle\overline{Q}_j Q^i\rangle$ and $\langle S\rangle$. (The genuine low energy Lagrangians in different vacua will be obtained below, both in the direct and dual theories).

It follows from (2.1) that there is a large number of various different vacua in this theory. But as for the realization of the global flavor symmetry $U(N_F)$, there are only two types of vacua: those with unbroken $U(N_F)$ and those with the spontaneous breaking $U(N_F) \to U(n_1) \times U(n_2)$, $n_1 + n_2 = N_F$.

As an example, we consider below only the br2-vacua (br=breaking) with $\langle(\overline{Q}Q)_2\rangle \gg \langle(\overline{Q}Q)_1\rangle$ and $n_2 > N_c$, $n_1 < \overline{N}_c$, and with the multiplicity $N_{\rm br2} = (\overline{N}_c - n_1) C^{n_1}_{N_F}$, $C^{n_1}_{N_F} = N_F!/(n_1!\, n_2!)$.

3 Fions $\Phi^{\rm i}_{\rm j}$ in the direct theory: one or three generations

At all scales $\mu < \Lambda_Q$ until the field Φ remains too heavy and non-dynamical (while the light quarks and gluons are still effectively massless and dynamical), i.e. until the perturbative running mass $\mu^{\rm pert}_{\Phi}(\mu) > \mu$, the field Φ decouples and can be integrated out, and the Lagrangian in the conformal regime takes the form at the scale $\mu \ll \Lambda_Q$ ($\overline{Q}_R, Q_R$ are renormalized fields)

$$K = z_Q(\Lambda_Q, \mu){\rm Tr}\Big(Q^\dagger Q + Q \to \overline{Q}\Big) = {\rm Tr}\left(Q^\dagger_R Q_R + (Q_R \to \overline{Q}_R)\right),\;\; z_Q(\Lambda_Q, \mu \ll \Lambda_Q) = \Big(\frac{\mu}{\Lambda_Q}\Big)^{\gamma_Q = \frac{3N_c - N_F}{N_F} > 0} \ll 1\,,$$

$$\mathcal{W}_Q = \frac{m_Q}{z_Q(\Lambda_Q, \mu)}{\rm Tr}(\overline{Q}_R Q_R) - \frac{1}{z^2_Q(\Lambda_Q, \mu) 2\mu_\Phi}\left({\rm Tr}\,(\overline{Q}_R Q_R)^2 - \frac{1}{N_c}\Big({\rm Tr}\,\overline{Q}_R Q_R\Big)^2\right). \tag{3.1}$$

Because the quark renormalization factor $z_Q(\Lambda_Q,\mu)$ decreases at smaller scale μ, it is seen from (3.1) that the role of the 4-quark term $\sim (\overline{Q}_R Q_R)^2$ increases with lowering energy. Hence, while it is irrelevant at the scale $\mu\sim\Lambda_Q$ because $\mu_\Phi\gg\Lambda_Q$, the question is whether it becomes dynamically relevant at some lower scale $\mu=\mu_o$. For this, we estimate the scale μ_o where this term becomes relevant in the conformal regime of the (effectively) massless theory of quarks and gluons:

$$\frac{\mu_o}{\mu_\Phi}\frac{1}{z^2_Q(\Lambda_Q,\mu_o)}=\frac{\mu_o}{\mu_\Phi}\Big(\frac{\Lambda_Q}{\mu_o}\Big)^{2\gamma_Q}\sim 1\quad\rightarrow\quad \mu_o\sim\Lambda_Q\Big(\frac{\Lambda_Q}{\mu_\Phi}\Big)^{\frac{1}{(2\gamma_Q-1)}}\sim\Lambda_Q\Big(\frac{\Lambda_Q}{\mu_\Phi}\Big)^{\frac{N_F}{3(2N_c-N_F)}>0}\ll\Lambda_Q\,. \tag{3.2}$$

We recall that even at those scales μ that the running mass of fions $\mu_\Phi(\mu)=\mu_\Phi/z_\Phi(\Lambda_Q,\mu)\gg\mu$ and so they are too heavy and dynamically irrelevant, **the quarks and gluons remain effectively massless and active**. Therefore, due to the Yukawa interactions of fions with quarks, the loops of still active light quarks (and gluons interacting with quarks) **still induce the power-like running renormalization factor** $z_\Phi(\Lambda_Q,\mu\ll\Lambda_Q)=(\mu/\Lambda_Q)^{\gamma_\Phi<0}\gg 1$ **of fions at all those scales until quarks are effectively massless**, i.e. $\mu>m_Q^{\rm pole}$ (see below).

It seems clear that the physical reason why the 4-quark terms in the superpotential (3.1) become relevant at scales $\mu < \mu_o$ is that **the fion field Φ which was too heavy and so dynamically irrelevant at $\mu > \mu_o$, $\mu_\Phi(\mu > \mu_o) > \mu$, becomes effectively massless at $\mu < \mu_o$, $\mu_\Phi(\mu < \mu_o) < \mu$, and begins to participate in the renorm-group evolution, i.e. it becomes relevant**. In other words, the four quark term in (3.1) 'remembers' about fions and signals about the scale below which the fions become effectively massless, $\mu_o = \mu_2^{\rm pole}(\Phi)$. This allows us to find the value of $z_\Phi(\Lambda_Q, \mu > \mu_o)$

$$\frac{\mu_\Phi}{z_\Phi(\Lambda_Q, \mu_o)} = \mu_o \quad \rightarrow \quad z_\Phi(\Lambda_Q, \mu_o < \mu < \Lambda_Q) = \left(\frac{\Lambda_Q}{\mu}\right)^{2\gamma_Q} \gg 1 \quad \rightarrow \quad \gamma_\Phi = -2\gamma_Q < 0\,. \tag{3.3}$$

Because the propagator of the renormalized fion fields look as $1/(p^2 - \mu_\Phi^2(p^2))$ and $|\mu_\Phi^2(p^2)| \lesseqgtr |p^2|$ at $p^2 \lesseqgtr \mu_o^2$, it is clear that there is a pole in the fion propagator at $p^2 = \mu_2^{\rm pole}(\Phi) = (\mu_o^2 - i\mu_o\Gamma_\Phi)$, i.e. **there is a second generation of all N_F^2 fields Φ_j^i** (the first one is at $\mu_1^{\rm pole}(\Phi) \gg \Lambda_Q$).

It can be shown that **the conformal regime remains the same** even at scales $m_Q^{\rm pole} < \mu < \mu_o$ where fion fields became relevant, and the quark and fion anomalous dimensions γ_Q and γ_Φ remain the same. I.e., the perturbative running mass $\mu_\Phi(\mu) \sim \mu_\Phi/z_\Phi(\Lambda_Q, \mu \ll \Lambda_Q) \ll \Lambda_Q$ of fions continues to decrease quickly with diminishing μ at all scales $m_Q^{\rm pole} < \mu < \Lambda_Q$ until quarks remain effectively massless, and becomes frozen only at scales below the quark physical mass $m_Q^{\rm pole}$, when the heavy quarks decouple (or are higgsed).

However, if $m_Q^{\rm pole} > \mu_o$, there is no pole in the fion propagator at scales $\mu < \Lambda_Q$. The reason is that quarks decouple as heavy at $\mu < m_Q^{\rm pole}$. And because $m_Q^{\rm pole} > \mu_o$, all fions Φ^i_j remain too heavy and irrelevant at this scale. Then, at $\mu < m_Q^{\rm pole}$, the running fion mass remains frozen at the large value $\mu_\Phi(\mu = m_Q^{\rm pole} > \mu_o) > m_Q^{\rm pole}$. The fions remain then dynamically irrelevant and unobservable as resonanses in this case at all scales $\mu < \Lambda_Q$.

But when $m_Q^{\rm pole} \ll \mu_o$, there will be not only the second generation of fions at $\mu = \mu_2^{\rm pole}(\Phi)$ but also **a third generation** at $\mu = \mu_3^{\rm pole}(\Phi) \ll \mu_2^{\rm pole}(\Phi)$. Indeed, after the heavy quarks decouple at the scale $m_Q^{\rm pole} \ll \mu_o$ and the renormalization factor $z_\Phi(\Lambda_Q, m_Q^{\rm pole})$ of fions becomes frozen **in the region of scales where the fions already became relevant**, the frozen value $\mu_\Phi(\mu < m_Q^{\rm pole}) = \mu_\Phi/z_\Phi(\Lambda_Q, \mu = m_Q^{\rm pole})$ of the fion mass is now: $\mu_\Phi(\mu = m_Q^{\rm pole}) \ll m_Q^{\rm pole}$. Therefore, **there is one more pole in the fion propagator** at $\mu = \mu_3^{\rm pole}(\Phi) = \mu_\Phi(\mu = m_Q^{\rm pole}) \ll m_Q^{\rm pole}$.

On the whole, in a few words for the direct theory.

a) The fions remain dynamically irrelevant and there are no poles in the fion propagator at scales $\mu < \Lambda_Q$ if $m_Q^{\rm pole} > \mu_o$.

b) If $m_Q^{\rm pole} < \mu_o \sim \Lambda_Q \Big(\frac{\Lambda_Q}{\mu_\Phi}\Big)^{\frac{N_F}{3(2N_c-N_F)}} \ll \Lambda_Q$, there are two poles in the fion propagator at scales $\mu \ll \Lambda_Q$: $\mu_2^{\rm pole}(\Phi) \approx \mu_o$ and $\mu_3^{\rm pole}(\Phi) \sim \mu_\Phi/z_\Phi(\Lambda_Q, m_Q^{\rm pole}) \ll \mu_2^{\rm pole}(\Phi)$. In other words, **the fions appear in three generations** in this case (we recall that there is always the largest pole mass of fions $\mu_1^{\rm pole}(\Phi) \gg \Lambda_Q$). Hence, the fions are effectively massless and dynamically relevant in the range of scales $\mu_3^{\rm pole}(\Phi) < \mu < \mu_2^{\rm pole}(\Phi)$.

Moreover, once the fions become effectively massless and dynamically relevant with respect to internal interactions, they begin to contribute simultaneously to the external anomalies (the 't Hooft triangles in the external background fields).

4 Mass spectra in br2 vacua. Direct theory, $\overline{\mathrm{b}}_{\mathrm{o}}/N_F = O(1)\,,\ (\overline{\mathrm{b}}_{\mathrm{o}} - 2n_1)\ >\ 0$

The general scheme for calculations of mass spectra both in the direct and dual theories looks as follows.

1) From the exact $\mathcal{W}_{\rm eff}$ in (2.1) the values of the quark and gluino condensates at $\mu = \Lambda_Q$, $\langle\overline{Q}Q\rangle$ and $\langle S\rangle$, can be found in each vacuum.

2) From this and from the knowledge of all anomalous dimensions in the conformal regime, all renormalization factors $z_i(\Lambda_Q, \mu < \Lambda_Q)$ for all fields in the Kahler terms are also known. Then the potentially possible values of pole masses of quarks, $m_Q^{\rm pole} = \langle m_Q^{\rm tot}\rangle / z_Q(\Lambda_Q, m_Q^{\rm pole})$, or possible gluon pole masses $(\mu_{gl}^{\rm pole})^2 \sim z_Q(\Lambda_Q, \mu_{gl})\langle\overline{Q}Q\rangle$ for higgsed quarks can be found (and similarly in the dual theory).

3) The hierarchies between them determine then the realized phase states and real mass spectra in each vacuum at given values of Lagrangian parameters.

From $\mathcal{W}_{\rm eff}$ in (2.1) the condensates of quarks in the direct theory look as

$$\langle(\overline{Q}Q)_2\rangle \sim m_Q\mu_\Phi,\;\; \langle(\overline{Q}Q)_1\rangle \sim \Lambda_Q^2\Big(\frac{\mu_\Phi}{\Lambda_Q}\Big)^{\frac{n_2}{n_2-N_c}}\Big(\frac{m_Q}{\Lambda_Q}\Big)^{\frac{N_c-n_1}{n_2-N_c}},\quad \frac{\langle(\overline{Q}Q)_1\rangle}{\langle(\overline{Q}Q)_2\rangle}\sim\Big(\frac{\mu_\Phi}{\mu_{\Phi,o}}\Big)^{\frac{N_c}{n_2-N_c}}\ll 1 \tag{4.1}$$

in br2 - vacua with $U(N_F)\to U(n_1)\times U(n_2),\; n_2>N_c\,, 1\leq n_1<\overline{N}_c\,.$ The largest among the masses smaller than Λ_Q are **masses of the N_F^2 second generation fions**, see (3.2),

$$\mu_2^{\rm pole}(\Phi_i^j)=\mu_o=\Lambda_Q\Big(\frac{\Lambda_Q}{\mu_\Phi}\Big)^{\frac{N_F}{3(2N_c-N_F)}}\ll\Lambda_Q\,,\quad i,j=1...N_F\,, \tag{4.2}$$

and **all N_F^2 fions become dynamically relevant at scales** $\mu<\mu_o$ (the cases when there are additional non-perturbative contributions to the masses of fions have to be considered separately, see below).

Some other possible characteristic masses look in this vacuum as

$$\langle m^{\rm tot}_{Q,1}\rangle=\frac{\langle(\overline{Q}Q)_2\rangle}{\mu_\Phi}\sim m_Q\,,\quad m^{\rm pole}_{Q,1}=\frac{\langle m^{\rm tot}_{Q,1}\rangle}{z_Q(\Lambda_Q,m^{\rm pole}_{Q,1})}\sim\Lambda_Q\Bigl(\frac{m_Q}{\Lambda_Q}\Bigr)^{N_F/3N_c}\gg m^{\rm pole}_{Q,2}\,, \tag{4.3}$$

$$\mu^2_{\rm gl,2}\sim z_Q(\Lambda_Q,\mu_{\rm gl,2})\langle(\overline{Q}Q)_2\rangle\gg\mu^2_{\rm gl,1}\,,\quad z_Q(\Lambda_Q,\mu_{\rm gl,2})\ll 1\quad\rightarrow\quad\mu_{\rm gl,2}\sim\Lambda_{YM}\ll m^{\rm pole}_{Q,1}. \tag{4.4}$$

where $m^{\rm pole}_{Q,1}$ and $m^{\rm pole}_{Q,2}$ are the pole masses of quarks $\overline{Q}_1, Q^1$ and $\overline{Q}_2, Q^2$ and $\mu_{\rm gl,1}$, $\mu_{\rm gl,2}$ are the gluon masses due to possible higgsing of these quarks. Hence, the largest mass is $m^{\rm pole}_{Q,1}$. **The overall phase is: all heavy quarks** (i.e. not higgsed but confined, $\langle Q^1\rangle=\langle Q^2\rangle=0$).

After the heaviest quarks $Q^1, \overline{Q}_1$ decoupled at $\mu < m^{\rm pole}_{Q,1}$, the lower energy theory has N_c colors and $N_F' = n_2 > N_c$ flavors of still active lighter quarks $\overline{Q}_2, Q^2$. In the range of scales $m^{\rm pole}_{Q,2} < \mu < m^{\rm pole}_{Q,1}$, it will remain in the conformal regime at $2n_1 < \overline{\rm b}_{\rm o},\ \overline{\rm b}_{\rm o} = (3\overline{N}_c - N_F) > 0$, while it will be not in the conformal but in the strong coupling regime at $2n_1 > \overline{\rm b}_{\rm o}$, with the gauge coupling $a(\mu \ll m^{\rm pole}_{Q,1}) = (m^{\rm pole}_{Q,1}/\mu)^{\nu > 0} \gg 1$. We do not consider the strong coupling regime here and for this reason we consider $2n_1 < \overline{\rm b}_{\rm o}$ only.

It follows from the exact $\mathcal{W}_{\rm eff}$ in (2.1) that the flavor symmetry is broken spontaneously in these br2 vacua as $U(N_F) \to U(n_1) \times U(n_2)$. It follows then from this that quarks $\overline{Q}_2, Q^2$ are not higgsed but confined. If they were higgsed, then $U(n_2)$ would be further broken spontaneously due to the rank condition because $n_2 > N_c$, this would contradict the exact (2.1). Therefore $m^{\rm pole}_{Q,2} = ({\rm several})\mu_{\rm gl,2}$, and the quarks $\overline{Q}_2, Q^2$ are not higgsed but confined. The confinement originates in this case from the $SU(N_c)$ $\mathcal{N} = 1$ SYM sector.

In the lower energy theory at $\mu < m^{\rm pole}_{Q,1}$ the pole mass of quarks $\overline{Q}_2, Q^2$ looks as

$$m^{\rm pole}_{Q,2} = \frac{m^{\rm pole}_{Q,1}}{z'_Q(m^{\rm pole}_{Q,1}, m^{\rm pole}_{Q,2})}\left(\frac{\langle(\overline{Q}Q)_1\rangle}{\langle(\overline{Q}Q)_2\rangle}\right) \sim ({\rm several})\Lambda_{\rm YM}, \quad {\rm z}'_{\rm Q}({\rm m}^{\rm pole}_{\rm Q,1}, {\rm m}^{\rm pole}_{\rm Q,2}) \sim \Big(\frac{{\rm m}^{\rm pole}_{\rm Q,2}}{{\rm m}^{\rm pole}_{\rm Q,1}}\Big)^{\frac{3{\rm N_c}-{\rm n_2}}{\rm n_2}} \ll 1\,. \tag{4.5}$$

Hence, after integrating out quarks $\overline{Q}_1, Q^1$ at $\mu < m^{\rm pole}_{Q,1}$ and then quarks $\overline{Q}_2, Q^2$ and $SU(N_c)$ gluons at $\mu < \Lambda^{(\rm br2)}_{YM}$ (through the Veneziano - Yankielowicz procedure), the Lagrangian of fions looks as, see (4.5),

$$K = z_{\Phi}(\Lambda_Q, m^{\rm pole}_{Q,1})\,{\rm Tr}\Big[\,\Phi^{\dagger}_{11}\Phi_{11} + \Phi^{\dagger}_{12}\Phi_{12} + \Phi^{\dagger}_{21}\Phi_{21} + z'_{\Phi}(m^{\rm pole}_{Q,1}, m^{\rm pole}_{Q,2})\Phi^{\dagger}_{22}\Phi_{22}\Big]\,, \tag{4.6}$$

$$z_{\Phi}(\Lambda_Q, m^{\rm pole}_{Q,1}) \sim \Big(\frac{\Lambda_Q}{m^{\rm pole}_{Q,1}}\Big)^{\frac{2(3N_c-N_F)}{N_F}} \gg 1, \quad \mathcal{W} = N_c S + \mathcal{W}_{\Phi}, \quad \mathcal{W}_{\Phi} = \frac{\mu_{\Phi}}{2}\Big({\rm Tr}\,(\Phi^2) - \frac{1}{\overline{N}_c}({\rm Tr}\,\Phi)^2\Big). \tag{4.7}$$

$$S = \Big(\Lambda^{\rm b_o}_Q \det m^{\rm tot}_Q\Big)^{1/N_c},\ m^{\rm tot}_Q = (m_Q - \Phi),\ \langle S\rangle = \big(\Lambda_{YM}\big)^3 \sim \Big(\frac{\mu_{\Phi}}{\Lambda_Q}\Big)^{\frac{n_2}{n_2-N_c}}\Big(\frac{m_Q}{\Lambda_Q}\Big)^{\frac{n_2-n_1}{n_2-N_c}}.$$

From (4.6),(4.7), the main contribution to the mass of **the third generation fions** Φ_{11} gives the term $\sim \mu_\Phi \Phi_{11}^2$,

$$\mu_3^{\rm pole}(\Phi_{11}) = \frac{\mu_\Phi}{z_\Phi(\Lambda_Q, m_{Q,1}^{\rm pole})} \sim \mu_\Phi \Big(\frac{m_Q}{\Lambda_Q}\Big)^{\frac{2(3N_c-N_F)}{3N_c}} \ll \Lambda_{YM}\,, \tag{4.8}$$

As for **the third generation fions** Φ_{22}, the main contribution to their masses comes from the non-perturbative term $\sim S$ in the superpotential (4.7)

$$\mu_3^{\rm pole}(\Phi_{22}) \sim \frac{\langle S\rangle}{\langle m_{Q,2}^{\rm tot}\rangle^2}\frac{1}{z_\Phi(\Lambda_Q, m_{Q,1}^{\rm pole}) z_\Phi^\prime(m_{Q,1}^{\rm pole}, m_{Q,2}^{\rm pole})} \sim m_{Q,2}^{\rm pole} \sim \Lambda_{YM}. \tag{4.9}$$

The third generation hybrid fions Φ_{12}, Φ_{21} **are massless**: $\mu_3^{\rm pole}(\Phi_{12}) = \mu_3^{\rm pole}(\Phi_{21}) = 0$, they are Nambu-Goldstone particles of the spontaneously broken global flavor symmetry: $U(N_F) \to U(n_1) \times U(n_2)$.

5 Mass spectra in br2 vacua. Dual theory, $\overline{\mathrm{b}}_{\mathrm{o}}/N_F = O(1)\,,\ (\overline{\mathrm{b}}_{\mathrm{o}} - 2n_1)\ >\ 0$

From $\mathcal{W}_{\text{eff}}$ in (2.1), in these vacua with $n_2 > N_c\,, 1 \leq n_1 < \overline{N}_c$ the condensates of mions and dual quarks look at $\mu = \Lambda_Q$ as

$$\langle M_2\rangle = \langle(\overline{Q}Q)_2\rangle \sim m_Q\mu_\Phi,\ \langle M_1\rangle = \langle(\overline{Q}Q)_1\rangle \sim \Lambda_Q^2\Big(\frac{\mu_\Phi}{\Lambda_Q}\Big)^{\frac{n_2}{n_2-N_c}}\Big(\frac{m_Q}{\Lambda_Q}\Big)^{\frac{N_c-n_1}{n_2-N_c}},\ \frac{\langle M_1\rangle}{\langle M_2\rangle} \sim \Big(\frac{\mu_\Phi}{\mu_{\Phi,\text{o}}}\Big)^{\frac{N_c}{n_2-N_c}} \ll 1\,,$$

$$\langle S\rangle = \frac{\langle M_1\rangle\langle M_2\rangle}{\mu_\Phi}, \quad \langle N_1\rangle = \langle(\overline{q}q)_1\rangle = \frac{\Lambda_Q\langle S\rangle}{\langle M_1\rangle} = \frac{\Lambda_Q\langle M_2\rangle}{\mu_\Phi} \sim m_Q\Lambda_Q \gg \langle(\overline{q}q)_2\rangle. \tag{5.1}$$

From these, the heaviest are N_F^2 mions M^i_j with the pole masses

$$\mu^{\text{pole}}(M^i_j) = \frac{\Lambda_Q^2/\mu_\Phi}{z_M(\Lambda_Q, \mu^{\text{pole}}(M))} \sim \Lambda_Q\Big(\frac{\Lambda_Q}{\mu_\Phi}\Big)^{\frac{N_F}{3(2N_c-N_F)}} \sim \mu_2^{\text{pole}}(\Phi^j_i) \gg \overline{\mu}^{\text{pole}}_{\text{gl},1}\,, \tag{5.2}$$

while some other possible characteristic masses look as

$$\overline{\mu}^{\rm pole}_{\rm gl,1}\sim\Lambda_Q\Big(\frac{\langle(\overline{q}q)_1\rangle}{\Lambda^2_Q}\Big)^{N_F/3N_c}\sim\Lambda_Q\Big(\frac{m_Q}{\Lambda_Q}\Big)^{N_F/3N_c}\sim m^{\rm pole}_{Q,1}\gg\overline{\mu}^{\rm pole}_{\rm gl,2}\,,\quad \overline{\mu}^{\rm pole}_{\rm gl,1}\gg\mu^{\rm pole}_{q,2}\gg\mu^{\rm pole}_{q,1}\,, \tag{5.3}$$

where $\overline{\mu}^{\rm pole}_{\rm gl,1,2}$ are the gluon masses due to possible higgsing of these quarks. Hence, the largest mass is $\overline{\mu}_{\rm gl,1}$ and the overall phase is $\mathbf{Higgs_1-Hq_2}$ (i.e. higgsed quarks q_1 and confined quarks q_2 with non-higgsed colors).

After integrating out all massive gluons and their scalar superpartners, the dual Lagrangian at $\mu = \overline{\mu}_{\rm gl,\,1}$ looks as

$$K = z_M(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) {\rm Tr}\, \frac{M^\dagger M}{\Lambda_Q^2} + z_q(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) {\rm Tr} \Big[\, 2\sqrt{N_{11}^\dagger N_{11}} + K_{\rm hybr} + \Big(\mathsf{q}_2^\dagger \mathsf{q}_2 + (\mathsf{q}_2 \to \overline{\mathsf{q}}_2) \Big) \Big]\,, \tag{5.4}$$

$$K_{\rm hybr} = \Bigg(N_{12}^\dagger \frac{1}{\sqrt{N_{11} N_{11}^\dagger}} N_{12} + (N_{12} \to N_{21}) \Bigg),\;\; z_q(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) = \Big(\frac{\overline{\mu}_{\rm gl,\,1}}{\Lambda_Q} \Big)^{\overline{\rm b}_{\rm o}/N_F}, \qquad z_M = z_q^{-2},\;\; \overline{\rm b}_{\rm o} = 3\overline{N}_c - N_F\,,$$

$$\overline{\mathcal{W}} = \Big[-\frac{2\pi}{\overline{\alpha}(\mu)} \overline{\mathsf{S}} \Big] - \frac{1}{\Lambda_Q} {\rm Tr} \Big(\overline{\mathsf{q}}_2 M_{22} \mathsf{q}_2 \Big) - \mathcal{W}_{MN} + \mathcal{W}_M, \tag{5.5}$$

$$\mathcal{W}_{MN} = \frac{1}{\Lambda_Q} {\rm Tr} \Big(M_{11} N_{11} + M_{21} N_{12} + N_{21} M_{12} + M_{22} N_{21} \frac{1}{N_{11}} N_{12} \Big),\;\; N_{12} = \langle \overline{q}_1 \rangle q_2,\;\; N_{21} = \overline{q}_2 \langle q_1 \rangle,$$

$$\mathcal{W}_M = m_Q {\rm Tr}\, M - \frac{1}{2\mu_\Phi} \Bigg[{\rm Tr}\, (M^2) - \frac{1}{N_c} ({\rm Tr}\, M)^2 \Bigg]\,,$$

where the nions (dual pions) N_{11} originate from higgsing of $\overline{q}^1, q_1$ dual quarks while the hybrid nions N_{12} and N_{21} are, in essence, the dual quarks $\overline{q}^2$, q_2 with higgsed colors. $\overline{\mathsf{q}}^2, \mathsf{q}_2$ are still active quarks $\overline{q}_2$, q_2 with non-higgsed colors. $\overline{\mathsf{S}}$ is the field strength squared of remained light dual $SU(\overline{N}_c - n_1)$ gluons.

The lower energy theory at $\mu < \overline{\mu}_{\rm gl,\,1}$ has $(\overline{N}_c - n_1)$ colors and $n_2 > N_c$ flavors, $0 < \overline{\rm b}^{\,\prime}_{\rm o} = (\overline{\rm b}_{\rm o} - 2n_1) < \overline{\rm b}_{\rm o}$. We consider here only the case $\overline{\rm b}^{\,\prime}_{\rm o} > 0$ when it remains in the conformal window. The fields N_{11}, N_{12}, N_{21} and M_{11}, M_{12}, M_{21} are frozen and do not evolve at $\mu < \overline{\mu}_{\rm gl,\,1}$, while the value of the pole mass $\mu^{\rm pole}_{q,2}$ in this lower energy theory is

$$\mu^{\rm pole}_{\mathsf{q},2} \sim \frac{\langle M_2\rangle}{\Lambda_Q}\frac{1}{z_q(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) z^{\prime}_q(\overline{\mu}_{\rm gl,\,1}, \mu^{\rm pole}_{\mathsf{q},2})} \sim ({\rm several})\Lambda_{\rm YM}\,, \quad {\rm z}^{\prime}_{\rm q}(\overline{\mu}_{\rm gl,\,1}, \mu^{\rm pole}_{\mathsf{q},2}) \sim \left(\frac{\mu^{\rm pole}_{\mathsf{q},2}}{\overline{\mu}_{\rm gl,\,1}}\right)^{\overline{\rm b}^{\prime}_{\rm o}/{\rm n}_2} \ll 1\,. \tag{5.6}$$

Finally, after integrating out remained non-higgsed (but confined) quarks $\overline{\mathsf{q}}^2, \mathsf{q}_2$ (confinement originates in this case from the $SU(\overline{N}_c - n_1)$ $\mathcal{N}=1$ SYM sector) as heavy ones and $\mathcal{N}=1$ $SU(\overline{N}_c - n_1)$ SYM gluons at $\mu < \Lambda_{YM}$, the lowest energy Lagrangian of mions and nions looks as, see (5.4),

$$K = z_M(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) {\rm Tr}\, K_M + z_q(\Lambda_Q, \overline{\mu}_{\rm gl,\,1}) \Big[\, 2\sqrt{N_{11}^{\dagger} N_{11}} + K_{\rm hybr} \Big], \tag{5.7}$$

$$K_M = \frac{1}{\Lambda_Q^2}\Big(M_{11}^{\dagger} M_{11} + M_{12}^{\dagger} M_{12} + M_{21}^{\dagger} M_{21} + z^{\prime}_M(\overline{\mu}_{\rm gl,\,1}, \mu^{\rm pole}_{\mathsf{q},2}) M_{22}^{\dagger} M_{22} \Big),\;\; z^{\prime}_M(\overline{\mu}_{\rm gl,\,1}, \mu^{\rm pole}_{\mathsf{q},2}) \sim \Big(\frac{\overline{\mu}_{\rm gl,\,1}}{\mu^{\rm pole}_{\mathsf{q},2}} \Big)^{\frac{2\overline{\rm b}^{\prime}_{\rm o}}{n_2}} \gg 1,$$

$$\mathcal{W} = -\overline{N}^{\,\prime}_c S - \mathcal{W}_{MN} + \mathcal{W}_M \,, \quad S = \Lambda^3_{YM} \left(\det \frac{\langle N_1 \rangle}{N_{11}} \det \frac{M_{22}}{\langle M_2 \rangle} \right)^{1/\overline{N}^{\,\prime}_c}, \quad \Lambda^3_{YM} \sim m_Q \langle M_1 \rangle .$$

From (5.7), the "masses" of mions look as

$$\mu(M_{11})\sim\mu(M_{12})\sim\mu(M_{21})\sim\frac{\Lambda_Q^2}{z_M(\Lambda_Q,\overline{\mu}_{\rm gl,\,1})\mu_\Phi}\sim\Big(\frac{\mu_{\Phi,\rm o}}{\mu_\Phi}\Big)\overline{\mu}_{\rm gl,\,1}\gg\overline{\mu}_{\rm gl,\,1}\,,\quad\frac{\mu(M_{11})}{\mu^{\rm pole}(M)}\ll 1\,, \tag{5.8}$$

$$\mu(M_{22})\sim\frac{\Lambda_Q^2}{z_M(\Lambda_Q,\overline{\mu}_{\rm gl,\,1})z^\prime_M(\overline{\mu}_{\rm gl,\,1},\mu^{\rm pole}_{\mathsf{q},2}))\mu_\Phi}\sim\Big(\frac{\mu_{\Phi,\rm o}}{\mu_\Phi}\Big)^{\frac{3N_c-n_2}{3(n_2-N_c)}}\overline{\mu}_{\rm gl,\,1}\gg\overline{\mu}_{\rm gl,\,1}\,,\quad\frac{\mu(M_{22})}{\mu(M_{11})}\ll 1\,, \tag{5.9}$$

while the pole masses of nions N_1^1 are

$$\mu^{\rm pole}(N_1^1)\sim\mu_\Phi\Big(\frac{m_Q}{\Lambda_Q}\Big)^{\frac{2(3N_c-N_F)}{3N_c}}\sim\mu_3^{\rm pole}(\Phi_1^1)\ll\Lambda_{YM}\,. \tag{5.10}$$

The hybrid nions N_{12}, N_{21} **are massless**: $\mu^{\rm pole}(N_{12})=\mu^{\rm pole}(N_{21})=0$, they are Nambu-Goldstone particles of the spontaneously broken global flavor symmetry: $U(N_F)\to U(n_1)\times U(n_2)$. The large mion "masses" (5.8),(5.9) are not their pole masses but simply the frozen values of their running masses: at the scale $\mu=\overline{\mu}_{\rm gl,\,1}$ for $\mu(M_{11})$ and at the scale $\mu=\mu^{\rm pole}_{\mathsf{q},2}$ for $\mu(M_{22})$. The reason is that all N_F^2 mion fields M^i_j are light and dynamically relevant only at scales $\mu^{\rm pole}(M)<\mu<\Lambda_Q$. They become too heavy and dynamically irrelevant at scales $\mu<\mu^{\rm pole}(M)$, see (5.2) The only pole masses of all N_F^2 mions M^i_j are $\mu^{\rm pole}(M)\sim\Lambda_Q\Big(\Lambda_Q/\mu_\Phi\Big)^{N_F/3(2N_c-N_F)}$ in (5.2).

6 Conclusions

A). The qualitatively new phenomenon was found in the direct theory due to the strong power-like renorm-group evolution in the conformal regime. - **The seemingly heavy and dynamically irrelevant N_F^2 fion fields Φ^i_j 'return back' and there appear two additional generations of light Φ-particles with small masses** $\mu_3^{\rm pole}(\Phi) \ll \mu_2^{\rm pole}(\Phi) \ll \Lambda_Q$. Moreover, the third generation fields Φ^1_2 and Φ^2_1 are massless, they are the Nambu-Goldstone particles of the spontaneously broken global flavor symmetry $U(N_F) \to U(n_1) \times U(n_2)$.

B). Let us compare now the mass spectra (for particle masses $M_k < \Lambda_Q$) in the direct theory and in its Seiberg dual one at $3N_c/2 < N_F < 2N_c$ and $\mu_\Phi < \mu_{\Phi,\rm o} = \Lambda_Q(\Lambda_Q/m_Q)^{(2N_c-N_F)/N_c}$.

$$\textbf{Part I:}\ \ \overline{\rm b}_{\rm o}/N_F = O(1)\,,\ (\overline{\rm b}_{\rm o} - 2n_1)\ >\ 0$$

1) The largest masses $\mu_2^{\rm pole}(\Phi^i_j) \sim \mu_o \sim \Lambda_Q(\Lambda_Q/\mu_\Phi)^{N_F/3(2N_c-N_F)}$ in the direct theory have N_F^2 second generation scalar fion superfields, and N_F^2 scalar mion superfields M^i_j with the same pole masses in the dual one (up to possible factors $O(1)$ which are hard to control).

Therefore, these two sets look undistinguishable (with our accuracy). It is also worth noting that when all N_F^2 fion fields Φ^i_j become relevant at $\mu < \mu_o$ in the direct theory, then all N_F^2 mion fields M^i_j become irrelevant in the dual one (and vice versa at $\mu > \mu_o$).

2) The next scale is $\Lambda_{YM} \ll m^{\rm pole}_{Q,1} \sim \overline{\mu}^{\rm pole}_{\rm gl,\,1} \sim \Lambda_Q(m_Q/\Lambda_Q)^{N_F/3N_c} \ll \mu^{\rm pole}_2(\Phi^i_j)$. Because all quarks with n_1 and n_2 flavors are confined in the direct theory and $m^{\rm pole}_{Q,1} \gg m^{\rm pole}_{Q,2}$, there are e.g.: a) **many** $SU(n_1)$ **adjoint in flavor quarkonia** $(\overline{Q}_1 Q^1)$ **with this scale of masses and with different spins and P and C-parities made from these quarks with** n_1 **flavors**, each adjoint multiplet with (n_1^2-1) equal mass particles; b) **many hybrid quarkonia like** $(\overline{Q}_1 Q^2) + (\overline{Q}_2 Q^1)$ **with such masses**, each multiplet with different spins and P and C-parities has the multiplicity $2n_1n_2$. On the other hand, in the dual theory with higgsed (i.e. not confined but screened) $\overline{\mathsf{q}}_1$ and q^1 dual quarks with such masses, there are **only fixed numbers of particles with fixed quantum numbers**: $n_1(2N_c - n_1)$ massive dual gluons and the same number of their scalar superpartners. **Therefore, the mass spectra at this scale are clearly distinguishable in the direct and dual theories**.

3) The next scale is $m^{\rm pole}_{Q,2} \sim \mu^{\rm pole}_{\mathsf{q},2} \sim \mu^{\rm pole}_3(\Phi^2_2) \sim \Lambda_{YM} \ll m^{\rm pole}_{Q,1}$. There are many gluonia in both direct and dual theories with such masses and it seems these can be undistinguishable. Besides, there are e.g. many $SU(n_2)$ adjoint in flavor quarkonia with different masses of this scale, with different spins and P and C-parities made from confined quarks $\overline{Q}_2, Q^2$ quarks in the direct theory, and from confined quarks $\overline{\mathsf{q}}^2, \mathsf{q}_2$ in the dual one. These two sets of quarkonia can also be undistinguishable. But there are additionally (n^2_2-1) elementary $SU(n_2)$ adjoint scalar superfields Φ^2_2 with such masses in the direct theory. And supposing naturally that the number of scalar quarkonia $(\overline{Q}_2 Q^2)$ and $(\overline{\mathsf{q}}^2\mathsf{q}_2)$ is the same in the direct and dual theories, these extra (n^2_2-1) elementary scalars Φ^2_2 will distinguish these two theories.

4) And finally for particles with nonzero masses, there are n^2_1 (i.e. (n^2_1-1) $SU(n_1)$ flavor adjoints plus one singlet) third generation lightest elementary scalar fields $(\Phi^{\rm pole}_3)^i_j,\ i,j=1...n_1$ with $\mu^{\rm pole}_3(\Phi^1_1) \ll \Lambda_{YM}$ in the direct theory and the same number and the same (up to possible factors $O(1)$) mass dual pions (nions) $N^i_j,\ i,j=1...n_1$ in the dual one. These two sets look undistinguishable (with our accuracy).

5) In the direct theory, $2n_1n_2$ fion fields Φ^1_2 and Φ^2_1 of the third generation and the same number of nions (dual pions) N^1_2 and N^2_1 in the dual theory are the Nambu-Goldstone particles of the spontaneously broken global flavor symmetry $U(N_F) \to U(n_1) \times U(n_2)$ and are all massless.

We conclude that, on the whole, the mass spectra of the direct and dual theories in this region are different (this is especially clearly seen in the point '2'), in disagreement with the Seiberg hypothesis about complete equivalence of such two theories.

Part II : $\overline{\mathrm{b}}_{\mathrm{o}}/N_F \ll 1\,,\ (\overline{\mathrm{b}}_{\mathrm{o}} - 2n_1)\ <\ 0$

There is the additional small parameter $0 < \overline{\mathrm{b}}_{\mathrm{o}}/N_F \ll 1$, $\overline{\mathrm{b}}_{\mathrm{o}} = (3\overline{N}_c - N_F) = (2N_F - 3N_c)$ and this allows to see **parametrical** differences between mass spectra of the direct and dual theories.

At these values of parameters, the qualitative difference is that regimes at $\mu < m^{\rm pole}_{Q,1}$ are not conformal now. The direct theory is in the strong coupling regime with $a(\mu \ll m^{\rm pole}_{Q,1}) \gg 1$, while the dual theory at $\mu < \overline{\mu}^{\rm pole}_{\rm gl,1}$ is in the weakly coupled infrared free logarithmic regime. Not going into details (and ignoring logarithmic factors of the dual theory RG-evolution at $\mu < \overline{\mu}^{\rm pole}_{\rm gl,1}$ for simplicity), we give only the results for this region of parameters.

A) Direct theory.

a) All N_F^2 masses of second generation mions $\mu_2^{\rm pole}(\Phi_i^j)=\mu_o$ remain the same as before.

b) The masses of $m_{Q,1}^{\rm pole}$ and $\mu^{\rm pole}(\Phi_1^1,\Phi_1^2,\Phi_2^1)$ are frozen at $\mu<m_{Q,1}^{\rm pole}$ and so remain the same as before.

c) The mass of $m_{Q,2}^{\rm pole}$ looks now as: $m_{Q,2}^{\rm pole}=(\mu_\Phi/\mu_{\Phi,\rm o})m_{Q,1}^{\rm pole}\gg\Lambda_{YM}$.

d) The mass $\mu_3^{\rm pole}(\Phi_2^2)$ is parametrically smaller now than before, it becomes the smallest nonzero mass among all others.

$$\mu_3^{\rm pole}(\Phi_2^2)\sim\left(\frac{\mu_\Phi}{\mu_{\Phi,\rm o}}\right)^{\frac{2n_1-\overline{\rm b}_{\rm o}}{n_2-N_c}>0}\mu_3^{\rm pole}(\Phi_1^1)\ll\mu_3^{\rm pole}(\Phi_1^1)\ll\Lambda_{YM}\,.$$

e) $2n_1n_2$ fion fields Φ_2^1 and Φ_1^2 of the third generation are massless as in the Part I above.

B) Dual theory.

a) All N_F^2 mion masses are parametrically smaller now than before:

$$\mu^{\rm pole}(M^i_j)\sim f_*^2 Z_q^2\,\mu_2^{\rm pole}(\Phi^j_i)\ll \mu_2^{\rm pole}(\Phi^j_i)\,,\quad Z_q\sim \exp\{-\frac{\overline{N}_c-n_1}{7\overline{\rm b}_{\rm o}}\}\ll 1\,,\quad f_*^2=O(\frac{\overline{\rm b}_{\rm o}}{N_F})\ll 1\,.$$

b) $\overline{\mu}^{\rm pole}_{\rm gl,1}$ is parametrically smaller now than before:

$$\overline{\mu}^{\rm pole}_{\rm gl,1}\sim Z_q^{1/2}\,m^{\rm pole}_{Q,1}\ll m^{\rm pole}_{Q,1}\,.$$

c) $\mu^{\rm pole}_{\mathsf{q},2}$ looks now as:

$$\mu^{\rm pole}_{\mathsf{q},2}\sim\frac{1}{Z_q}\Bigl(\frac{\mu_{\Phi,{\rm o}}}{\mu_\Phi}\Bigr)^{\frac{2n_1-\overline{\rm b}_{\rm o}}{3(n_2-N_c)}>0}\Lambda_{YM}\gg\Lambda_{YM}\,,\quad \mu^{\rm pole}_{\mathsf{q},2}\sim\frac{1}{Z_q}\,m^{\rm pole}_{Q,2}\gg m^{\rm pole}_{Q,2}\gg\Lambda_{YM}\,.$$

Both direct quarks $\overline{Q}_2$, Q_2 and dual ones $\overline{\mathsf{q}}_2$, q_2 are weakly confined (i.e. the string tension originating from SYM is much smaller than quark masses, $\sigma^{1/2} \sim \Lambda_{YM} \ll m^{\rm pole}_{Q,2} \ll \mu^{\rm pole}_{\mathsf{q},2}$) and form a large number of various quarkonia. But quarks $\overline{\mathsf{q}}_2$, q_2 are non-relativistic and weakly coupled in the dual theory, so that the mass splittings between adjacent levels of dual quarkonia are parametrically small, $\delta M/M \sim O(\overline{\rm b}_{\rm o}^{\,2}/N_F^2) \ll 1$, while there is nothing similar in the strongly coupled direct theory.

d) $\mu^{\rm pole}(N_1^1)$ is parametrically larger now than before

$$\mu^{\rm pole}(N_1^1) \sim \frac{1}{Z_q}\mu_3^{\rm pole}(\Phi_1^1) \gg \mu_3^{\rm pole}(\Phi_1^1)\,.$$

e) $2n_1 n_2$ fion fields N_2^1 and N_1^2 are massless as in the Part I above.

It is seen that at $\overline{\mathrm{b}}_{\mathrm{o}}/N_F \ll 1$ in this Part II, **in addition to clear qualitative differences in point '2' of the Part I above**, all corresponding mass scales of the direct and dual theories in this region are **parametrically different** (and logarithmic factors present in the dual theory result in additional parametrical differences of corresponding masses). Therefore, there are no reasons for them to become exactly equal at $\overline{\mathrm{b}}_{\mathrm{o}}/N_F = O(1)$ in the Part I above.

On the whole, we conclude that, although similar in a number of respects, the direct and Seiberg's dual $\mathcal{N}=1$ SQCD-like theories have different mass spectra and are not equivalent. As was shown above, this is clearly seen at $(3\overline{N}_c - N_F)/N_F \ll 1$ where the corresponding masses are **parametrically different**. (And similarly at both ends of the conformal window, i.e. at $(3\overline{N}_c - N_F)/N_F \ll 1$ or $(3N_c - N_F)/N_F \ll 1$ in the standard $\mathcal{N}=1$ SQCD and its Seiberg's dual, both without fields Φ).

Much more examples can be found in: V.L. Chernyak, arXiv:1906.08643, pp.1-54 [hep-th]