Three-Reggeon cuts in QCD amplitudes

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Novosibirsk

Session-conference of the Section of Nuclear Physics DPS RAS
March 10-12 2020 Academicpark, Novosibirsk, Russia

Contents

Introduction

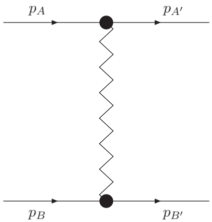
One of remarkable properties of QCD is the Reggeization of all elementary particles in perturbation theory, which is very important for theoretical description of high energy processes. The gluon Reggeization is especially important because it determines the high energy behaviour of non-decreasing with energy cross sections. In particular, it appears to be the basis of the BFKL (Balitskii-Fadin-Kuraev-Lipatov) equation, which was first derived in non-Abelian theories with spontaneously broken symmetry
F. V.S., Kuraev E.A., Lipatov L.N., 1975
and whose applicability in QCD was then shown
Balitsky I.I., Lipatov L.N., 1978.
The equation was derived using unitarity and analyticity.

Introduction

In each order of perturbation theory dominant (having the largest $\ln s$ degrees) are amplitudes with gluon quantum numbers and negative signature (parity with respect to $s \to u \simeq -s$). They determine the s-channel discontinuities of amplitudes with the same and all other possible quantum numbers.
It is extremely important that both in the leading logarithmic approximation (LLA) and in the next-to-leading one (NLLA) the amplitudes used in the unitarity relations are determined by the Regge pole contributions and have a simple factorized form (pole Regge form). Due to this, the Reggeization provides a simple derivation of the BFKL equation in the LLA and in the NLLA.
But the Regge pole contributions are not sufficient in the NNLLA.
The deviations can be explained by the three-Reggeon cuts.

Pole Regge form of QCD amplitudes and its violation

For elastic scattering processes $A + B \to A' + B'$ in the Regge kinematical region: $s \simeq -u \to \infty$, t fixed (i.e. not growing with s) the Reggeization means that scattering amplitudes with the gluon quantum numbers in the t-channel and negative signature (symmetry with respect to $s \leftrightarrow u$) is written as



$$\mathcal{A}_{AB}^{A'B'} = \Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)}\right]\Gamma^c_{B'B}\ ;$$

Pole Regge form of QCD amplitudes and its violation

$\Gamma^{c}_{P'P}$–particle-particle-Reggeon (PPR) vertices or scattering vertices ("c" are colour indices); $j(t) = 1 + \omega(t)$ – Reggeon trajectory.

Besides the elastic amplitudes, unitarity relations contain an infinite number of inelastic amplitudes.

They must be known in the kinematics giving the main contributions – the multi-Regge kinematics (MRK).

The Reggeization provides a definite form not only of elastic amplitudes, but of inelastic amplitudes in the MRK as well.

The important property of Regge poles is the factorization of their contributions

$$\mathcal{A}_{gg}^{g'g'}\mathcal{A}_{qq}^{q'q'} = \left(\mathcal{A}_{gq}^{g'q'}\right)^2$$

Pole Regge form of QCD amplitudes and its violation

Validity of the pole Regge form is proved now in all orders of perturbation theory in the coupling constant g both in the LLA and in the NLLA.

The pole Regge form is violated in the NNLLA.

The first observation of the violation was done

Del Duca V., Glover E.W.N., 2001

at consideration of the high-energy limit of the two-loop amplitudes for gg, gq and qq scattering. The discrepancy appears in non-logarithmic terms.

If the pole Regge form would be correct in the NNLLA, they should satisfy the factorization condition. However, it is not the case.

Using the infrared factorization techniques, consideration of the terms responsible for breaking of the pole Regge form in amplitudes of elastic scattering in QCD was performed by

Del Duca V., Falcioni G., Magnea L., Vernazza L., 2013-2015.

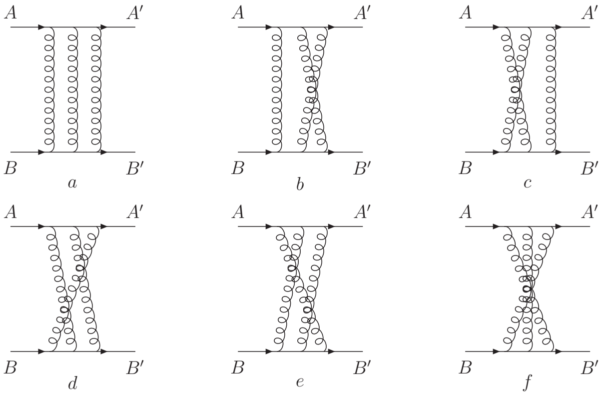
In particular, the non-logarithmic terms not satisfying the factorization condition at two-loops were recovered and single-logarithmic terms at three loops violating the pole Regge form were found.
It is necessary to say that, in general, breaking the pole Regge form is not a surprise.
It is well known that Regge poles in the complex angular momenta plane generate Regge cuts. Moreover, in amplitudes with positive signature the Regge cuts appear already in the LLA. In particular, the BFKL Pomeron is the two-Reggeon cut in the complex angular momenta plane. But in amplitudes with negative signature Regge cuts must be at least three-Reggeon ones and can appear only in the NNLLA.

Pole Regge form of QCD amplitudes and its violation

It was natural to expect that the observed violation of the pole Regge form can be explained by existence of the cut.

Indeed, all known cases of breaking of the pole Regge form are now explained by the three-Reggeon cuts
F. V.S., 2016; F. V.S., Lipatov L.N., 2017,
Caron-Huot S., Gardi E., Vernazza L., 2017
It could be considered as a good new if the explanations were the same or at least similar. Unfortunately, it is not so. The differences in the explanations start from the approaches used. The approach used in the first case can be called diagrammatic, since it starts from Feynman diagrams. Contrary, the second approach has no relation to Feynman diagrams and is based on Wilson line representation of high energy scattering amplitudes.
Both approaches explain the violation of the pole form in three loops but in different ways.

Due to the signature conservation the cut with negative signature must be the three-Reggeon one.
Since our Reggeon is the Reggeized gluon, the three-Reggeon cut first contribute to amplitudes corresponding to the diagrams



Diagrammatic approach. Appearance of the cut

The amplitude of the process $\mathcal{A}_{AB}^{A'B'}$ can be written as the sum over permutations σ of products of colour factors and colour-independent matrix elements:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\sigma} \left(C_{AB}^{(0)\sigma} \right)_{\alpha\beta}^{\alpha'\beta'} M_{AB}^{(0)\sigma}(s,t) \ ,$$

where α and β (α' and β') are colour indices of incoming (outgoing) projectile A and target B respectively. We use the same letters for quark and gluon colour indices; it should be remembered, however, that there is no difference between upper and lower indices (running from 1 to $N_c^2 - 1$) for gluons, whereas for quarks lower and upper indices (running from 1 to N_c) refer to mutually related representations.
The colour factors can be decomposed into irreducible representations $\mathcal{R}$ of the colour group in the t-channel:

Diagrammatic approach. Appearance of the cut

$$\left(C_{AB}^{(0)\sigma}\right)^{\alpha'\beta'}_{\alpha\beta} = \sum_R [\mathcal{P}_{AB}^R]^{\alpha'\beta'}_{\alpha\beta} \sum_\sigma \mathcal{G}(R)_{AB}^{(0)\sigma},$$

where

$$[\mathcal{P}_{AB}^R]^{\alpha'\beta'}_{\alpha\beta} = \sum_n [\mathcal{P}_A^{R,n}]^{\alpha'}_{\alpha} [\mathcal{P}_B^{R,n}]^{\beta'}_{\beta}$$

$\hat{\mathcal{P}}^{R,n}$ is the projection operator on the state n in the representation $\mathcal{R}$,

$$\mathcal{G}(R)_{AB}^{(0)\sigma} = \frac{1}{N_R \; T_A \; T_B} \left(\mathcal{T}_A^{c_1} \mathcal{T}_A^{c_2} \mathcal{T}_A^{c_3}\right)^{\alpha'}_{\alpha}$$

$$\times \left(\mathcal{T}_B^{c_1^\sigma} \mathcal{T}_B^{c_2^\sigma} \mathcal{T}_B^{c_3^\sigma}\right)^{\beta'}_{\beta} [\mathcal{P}_{AB}^R]^{\alpha\beta}_{\alpha'\beta'} ,$$

N_R is the dimension of the representation R, $\mathcal{T}^a$ are the colour group generators in the corresponding representations,

Diagrammatic approach. Appearance of the cut

$[\mathcal{T}^a, \mathcal{T}^b] = i f_{abc} \mathcal{T}^c$; for all representations,
$(\mathcal{T}^a)^b{}_c = -i f_{abc}$ for gluons,
$(\mathcal{T}^a)^{\alpha'}{}_\alpha = (t^a)^{\alpha'}{}_\alpha$ for quarks;
$\mathrm{Tr}(\mathcal{T}^a_A \mathcal{T}^b_B) = T_A \delta_{AB} \delta_{ab}, \quad T_q = 1/2, \quad T_g = N_c$.

In contrast to the Reggeon, which contributes only to amplitudes with the adjoint representation of the colour group (colour octet in QCD) in the *t*-channel, the cut can contribute to various representations.

Possible representations for quark-quark and quark-gluon scattering are only singlet (**1**) and octet (**8**), whereas for the gluon-gluon scattering there are also **10**, **10*** and **27**.

Additional restrictions are imposed by signature.

Taking into account Bose statistic for gluons, symmetry of the representations **1** and **27**, antisymmetry **10** and **10*** and existence both symmetric $\mathbf{8_s}$ and antisymmetric $\mathbf{8_a}$ representations for them, gives that

besides the Reggeon channel **8**

amplitudes with negative signature there are in the representations
1 for quark-quark-scattering
and in the representation
10 and **10*** for the gluon-gluon scattering.
The important thing is the existence itself of the representations of the colour group other than the adjoint one in amplitudes with negative signature, which means existence of singularities different from the Regge pole in the complex momenta plane.
Corresponding projection operators are

$$[\mathcal{P}_{gg}^{\mathbf{8_a}}]^{ab}_{a'b'} = \frac{f_{aa'c}f_{bb'c}}{N_C} \ ,$$

$$[\mathcal{P}_{gg}^{\mathbf{10}}]^{ab}_{a'b''} = \frac{1}{4}\Big(\delta_{ab}\delta_{a'b'} - \delta_{ab'}\delta_{a'b} - 2\frac{f_{aa'c}f_{bb'c}}{N_C}$$

$$+if_{ba'c}d_{b'ac} + if_{ba'c}d_{b'ac}\Big) \ , \quad [\mathcal{P}_{gg}^{\mathbf{10}^*}]^{ab}_{a'b''} = \Big([\mathcal{P}_{gg}^{\mathbf{10}}]^{ab}_{a'b''}\Big)^* \ ,$$

Diagrammatic approach. Appearance of the cut

$$[\mathcal{P}^{\mathbf{8}}_{qq}]^{\alpha'\beta'}_{\alpha\beta} = 2(t^c)^{\alpha'}_{\alpha}(t^c)^{\beta'}_{\beta} \,,$$

$$[\mathcal{P}^{\mathbf{1}}_{qq}]^{\alpha'\beta'}_{\alpha\beta} = \sqrt{\frac{2}{N_c}}\delta^{\alpha'}_{\alpha}\delta^{\beta'}_{\beta} \,,$$

$$[\mathcal{P}^{\mathbf{8_a}}_{gq}]^{a'\beta'}_{a\beta} = \frac{1}{N_c} f^{c}_{aa'}(t^c)^{\beta'}_{\beta} \,,$$

It turns out that for the representations R different from the Reggeized gluon one the colour coefficients $\mathcal{G}(R)^{(0)\sigma}_{AB}$ do not depend on σ:

$$\mathcal{G}(\mathbf{10}+\mathbf{10}^*)^{(0)\sigma}_{gg} = -\frac{3}{4} \,, \quad \mathcal{G}(\mathbf{1})^{(0)\sigma}_{qq} = \frac{(N_c^2-4)(N_c^2-1)}{8N_c^2} \,,$$

so that momentum dependent factors for them summed up to the eikonal amplitude

$$\sum M^{(0)\sigma}_{AB}(s,t) = A^{eik} = g^6\frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\vec{q}^{\,2}A_2(q_\perp) \,,$$

where $A_2(q_\perp)$ is depicted by the diagram

and is written as

$$A_2(q_\perp) = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2}{(2\pi)^{2(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} (\vec{q} - \vec{l}_1 - \vec{l}_2)^2} \, .$$

Note that the "infrared" ϵ is used, $\epsilon = (D-4)/2$, D is the space-time dimension.

This result is very important, because contribution of the cut must be gauge invariant, whereas $M_{AB}^{(0)\sigma}$ taken separately are gauge dependent.

In the Reggeized gluon channel the colour coefficients $\mathcal{G}(R)^{(0)\sigma}_{AB}$ depend on σ. However, this dependence has a specific form

$$\mathcal{G}(\mathbf{8_a})^{(0)b}_{AB} = \mathcal{G}(\mathbf{8_a})^{(0)c}_{AB} = \mathcal{G}(\mathbf{8_a})^{(0)d}_{AB} = \mathcal{G}(\mathbf{8_a})^{(0)e}_{AB} \equiv \mathcal{G}(\mathbf{8_a})^{(0)}_{AB} ,$$

$$\mathcal{G}(\mathbf{8_a})^{(0)a}_{gg} = \mathcal{G}(\mathbf{8_a})^{(0)f}_{gg} , \quad \mathcal{G}(\mathbf{8_a})^{(0)a}_{gq} = \mathcal{G}(\mathbf{8_a})^{(0)f}_{gq} ,$$

$$\frac{1}{2}\left[\mathcal{G}(\mathbf{8_a})^{(0)a}_{AB} + \mathcal{G}(\mathbf{8_a})^{(0)f}_{AB}\right] = \mathcal{G}(\mathbf{8_a})^{(0)}_{AB} + \frac{N_c^2}{8} .$$

Since

$$\mathcal{G}(\mathbf{8})^{(0)}_{gg} = \frac{3}{2} , \quad \mathcal{G}(\mathbf{8})^{(0)}_{gq} = \frac{1}{4} , \quad \mathcal{G}(\mathbf{8})^{(0)}_{qq} = \frac{1}{4}\left(-1 + \frac{3}{N_c^2}\right) ,$$

it is evident that the pole factorization is violated.
But it is seen also that the terms violating the pole factorization have σ-independent colour coefficients, so that momentum factors for them summed up to the eikonal amplitude.

Separation of the pole and cut contributions is impossible in the two-loop approximation because of the ambiguity of the allocation of the part of the amplitudes violating the factorization. The separation becomes possible in higher loops, due to different energy dependence of the pole and cut contributions. Energy dependence of the pole contribution is determined by the Regge factor of the Reggeized gluon $\exp(\omega(t)\ln s)$, where $\omega(t)$ is the gluon trajectory, whereas for the three-Reggeon cut it is

$$e^{[(\hat{\omega}_1+\hat{\omega}_2+\hat{\omega}_3+\hat{\mathcal{K}}_r(1,2)+\hat{\mathcal{K}}_r(1,3)+\hat{\mathcal{K}}_r(2,3))\ln s]}\ ,$$

where $\hat{\mathcal{K}}_r(m,n)$ is the real part of the BFKL kernel describing interaction between Reggeons m and n.

With the help of the integral representation of the trajectory

$$\omega(t) = -g^2 N_c \vec{q}^{\,2} \int \frac{d^{2+2\epsilon} l}{2(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{q}-\vec{l})^2}$$

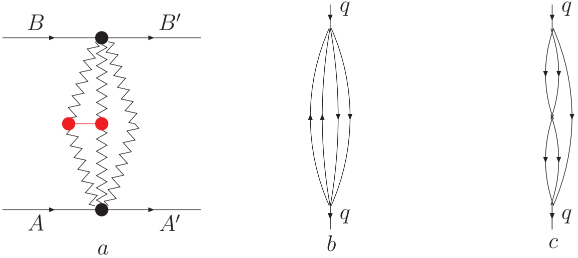
and the explicit form of the real part of the kernel describing interaction between two Reggeons with transverse momenta $\vec{l}_1$ and $\vec{l}_2$ and colour indices c_1 and c_2

$$\left[\mathcal{K}_r(\vec{q}_1, \vec{q}_2; \vec{k})\right]_{c_1 c_2}^{c_1' c_2'} = -T^a_{c_1 c_1'} T^a_{c_2 c_2'} \frac{g^2}{(2\pi)^{D-1}} \left[\frac{\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\,2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right],$$

where $\vec{q}_1 + \vec{q}_2 = \vec{q}_1' + \vec{q}_2' = \vec{q}\,, \;\; \vec{q}_1 - \vec{q}_1' = \vec{q}_2' - \vec{q}_2 = \vec{k},$

Diagrammatic approach. Three loops

the first order corrections are expressed through the diagrams b and c.



It turns out that the first order colour coefficient, $\mathcal{G}(\mathbf{R})^{(n)\sigma}_{AB}$ is simply proportional to $\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB}$, so that the three-loop correction is

$$\mathcal{G}(\mathbf{R})_{AB}^{(0)(cut)}\; g^8 \frac{s}{t}\left(\frac{-4\pi^2}{3}\right)\vec{q}^{\,2}\left(\left(\frac{3}{2}N_c - C_2(\mathbf{8})\right)A_3^b(q_\perp)\right.$$
$$\left. -\frac{1}{2}\left(3N_c - C_2(\mathbf{8})\right)A_3^c(q_\perp)\right)\;\ln s\;,$$

where

$$A_3^b(q_\perp) = -\int \frac{d^{2+2\epsilon}l_1\, d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3}{(2\pi)^{3(3+2\epsilon)}\vec{l}_1^{\,2}\vec{l}_2^{\,2}\vec{l}_3^{\,2}(\vec{q}-\vec{l}_1-\vec{l}_2-\vec{l}_3)^2}\;,$$

$$A_3^c(q_\perp) = \int \frac{d^{2+2\epsilon}l_1\, d^{2+2\epsilon}l_2 d^{2+2\epsilon}l_3(\vec{q}-l_1)^2}{(2\pi)^{3(3+2\epsilon)}\vec{l}_1^{\,2}\vec{l}_2^{\,2}\vec{l}_3^{\,2}(\vec{q}-\vec{l}_1-\vec{l}_2)^2(\vec{q}-\vec{l}_1-\vec{l}_3)^2}\;.$$

The calculation of the three-loop corrections showed that the violation of the pole Regge form, analysed in this approximation with the help of the infrared factorization, can be explained by the pole and cut contributions. In other words, the restrictions imposed by the infrared factorization on the parton scattering amplitudes with the adjoint representation of the colour group in the *t*-channel and negative signature can be fulfilled in the NNLLA at two and three loops if besides the Regge pole contribution there is the Regge cut contribution

$$\mathcal{G}(\mathbf{8_a})^{(0)(cut)}_{AB} g^6 \frac{s}{t} \left(\frac{-4\pi^2}{3} \right) \vec{q}^{\,2}$$

$$\times \left(A_2(q_\perp) + g^2 N_c \ln s \left(\frac{1}{2} A_3^b(q_\perp) - A_3^c(q_\perp) \right) \right) \; ,$$

$$\mathcal{G}(\mathbf{8_a})^{(0)(cut)}_{gg} = -\frac{3}{2} \, , \quad \mathcal{G}(\mathbf{8_a})^{(0)(cut)}_{gq} = - - \frac{3}{2} \, ,$$

$$\mathcal{G}(\mathbf{8_a})^{(0)(cut)}_{qq} = \frac{3(1 - N_c^2)}{4N_c^2} \, .$$

Wilson line approach

The explanation of the violation of the pole Regge form given in Caron-Huot S., Gardi E., Vernazza L., 2017
differs from described above. In this paper, no connection of the three-Reggeon cuts with Feynman diagrams was traced. The colour coefficients $\mathcal{G}(\mathbf{R})^{(0)C}_{AB}$ for the cut contributions are taken as

$$\mathcal{G}(\mathbf{R})^{(0)C}_{AB} = \frac{1}{6! N_{\mathbf{R}}\ T_A\, T_B} \left(\mathcal{T}_A^{c_1}\mathcal{T}_A^{c_2}\mathcal{T}_A^{c_3}\right)^{\alpha'}_{\ \alpha} \left(\sum_{\sigma} \mathcal{T}_B^{c_1^\sigma}\mathcal{T}_B^{c_2^\sigma}\mathcal{T}_B^{c_3^\sigma}\right)^{\beta'}_{\ \beta} [\mathcal{P}^{\mathbf{R}\,*}_{AB}]^{\alpha\beta}_{\ \alpha'\beta'} \ .$$

As for the momentum dependent part, it is taken equal to A^{eik}. For the representations **R** different from the Reggeized gluon one it agrees with the diagrammatic approach, since the colour coefficients $\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB}$ do not depend on σ for such representations. Therefore, the cut contributions in both approaches are the same for these representations.

Wilson line approach

But it is not so for the adjoint representation, where in the Wilson line approach the colour coefficients $\mathcal{G}(\mathbf{8})^{(0)C}_{AB}$ turn out to be

$$\mathcal{G}(\mathbf{8})^{(0)C}_{AB} = \mathcal{G}(\mathbf{8})^{(0)}_{AB} + \frac{N_c^2}{24} \ .$$

As for the momentum dependent part, it is also taken equal to A^{eik}. It looks strange from the point of view of the diagrammatic approach, because appearance of A^{eik} requires equality of all colour coefficients.

In two loops difference Δ_{AB} between two approaches is such that

$$\Delta_{gg} + \Delta_{qq} = 2\Delta_{gq}$$

and therefore it can be attributed to the pole contribution. To do this, it is sufficient to change the two-loop contributions to the gluon-gluon-Reggeon and quark-quark-Reggeon vertices.

But in three loops it turned out impossible to explain the violation of the pole form only by the cut. It becomes possible introducing the Reggeon-cut mixing with the colour coefficients

$$\mathcal{G}(\mathbf{8})^{(0)mix}_{AB} = \frac{1}{6(N_c^2-1)\ T_A\, T_B}\sum_{\sigma} Tr\left(\mathcal{T}_g^{c}\mathcal{T}_g^{c_1^\sigma}\mathcal{T}_g^{c_2^\sigma}\mathcal{T}_g^{c_3^\sigma}\right)$$

$$\times\left[\mathcal{T}_A Tr\left(\mathcal{T}_B^{c}\mathcal{T}_B^{c_1}\mathcal{T}_B^{c_2}\mathcal{T}_B^{c_3}\right)+\mathcal{T}_B Tr\left(\mathcal{T}_A^{c}\mathcal{T}_A^{c_1}\mathcal{T}_A^{c_2}\mathcal{T}_A^{c_3}\right)\right] .$$

The mixing contributes only in three loops.

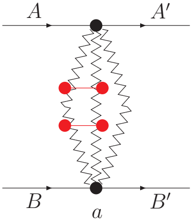
It is necessary to note that in the Wilson line approach the cut contribution is not suppressed at large N_c, i.e. it exists in the planar $N = 4$ SYM, in contradiction with the common opinion, that in the high energy limit the four-point amplitudes in this theory are given by the Reggeized gluon contribution.

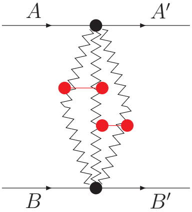
In the diagrammatic approach, the Reggeon-cut mixing is not necessary in the three-loop approximation.

Whether mixing is necessary can be verified in the four-loop approximation.

Four loops

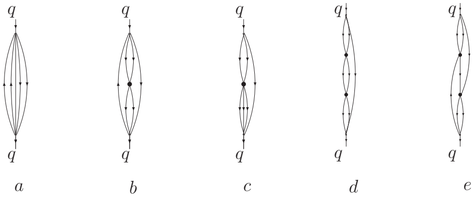
In the four loops there are three types of corrections. The first (simplest) ones come from account of the Regge factors of each of three Reggeons. The second type of the corrections are given by the products of the trajectories and real parts of the BFKL kernel, and the third come from account of Reggeon-Reggeon interactions.





Four loops

All the types of the corrections are expressed through the integrals in the transverse momentum space corresponding to the diagrams



Four loops

$$I_i = \int \frac{d^{2+2\epsilon} l_1 \, d^{2+2\epsilon} l_2 \, d^{2+2\epsilon} l_3}{(2\pi)^{3(3+2\epsilon)} \vec{l}_1^{\,2} \vec{l}_2^{\,2} \vec{l}_3^{\,2}} F_i \delta^{3+2\epsilon}(\vec{q} - \vec{l}_1 - \vec{l}_2 - \vec{l}_3) \; ,$$

$$F_a = f_1(\vec{l}_1) f_1(\vec{l}_2) \; , \quad F_b = f_1(\vec{l}_1) f_1(\vec{l}_1) \; , \quad F_c = f_2(\vec{l}_1 + \vec{l}_2) \; ,$$

$$F_d = f_1(\vec{l}_1 + \vec{l}_2) f_1(\vec{l}_1 + \vec{l}_2) \; , \quad F_e = f_1(\vec{q} - \vec{l}_1) f_1(\vec{q} - \vec{l}_3) \; ,$$

$$f_1(\vec{k}) = \vec{k}^{\,2} \int \frac{d^{2+2\epsilon} l}{(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{l} - \vec{k})^2} \; , \quad f_2(\vec{k}) = \int \frac{d^{2+2\epsilon} l f_1(\vec{l})}{(2\pi)^{(3+2\epsilon)} \vec{l}^{\,2} (\vec{l} - \vec{k})^2} \; .$$

These integrals enter in the total four-loop correction with different colour factors in the approaches with or without the Reggeon-cut mixing.

Four loops

Calculation of the colour factors $\mathcal{G}(\mathbf{R})^{(2)\sigma}_{AB}$ is not so simple as for three loops.

Of course, it is trivial for the squared virtual part contributions. The corresponding colour factor is

$$\mathcal{G}(R)^{(2)\sigma}_{AB,VV} = \frac{N_c^2}{4}\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB} \ .$$

It is not difficult also for the products of the virtual and real parts,

$$\mathcal{G}(R)^{(2)\sigma}_{AB,VR} = \frac{N_c}{2}\left(C_2(\mathbf{R}) - 3N_c\right)\mathcal{G}(\mathbf{R})^{(0)\sigma}_{AB} \ .$$

However, it is not so simple for the squared real part. It contains two types of matrix elements

$$\mathcal{G}(\mathbf{R})^{(s)\sigma}_{AB} = \langle\Psi^{\sigma}_{B}|\sum_{i\neq j=1}^{3}\hat{T}^c(i)\hat{T}^c(j)\hat{T}^d(i)\hat{T}^d(j)|\Psi_A\rangle$$

Four loops

and

$$\mathcal{G}(\mathbf{R})_{AB}^{(v)\sigma} = \langle\Psi_B^\sigma| \sum_{i\neq j\neq k=1}^{3} \hat{T}^c(i)\hat{T}^c(j)\hat{T}^d(i)\hat{T}^d(k)|\Psi_A\rangle \; .$$

Due to the colour conservation

$$\mathcal{G}(\mathbf{R})_{AB}^{(s)\sigma} + \mathcal{G}(\mathbf{R})_{AB}^{(v)\sigma}\frac{1}{4}\left(C_2(\mathbf{R}) - 3N_c\right)^2 \mathcal{G}(\mathbf{R})_{AB}^{(0)\sigma} \; .$$

The calculations give:

$$\mathcal{G}(\mathbf{1})_{qq}^{(s)} = \frac{3}{4}N_c^2\mathcal{G}(\mathbf{1})_{qq}^{(0)} \; , \;\; \mathcal{G}(\mathbf{1})_{qq}^{(s)} = \frac{3}{2}N_c^2\mathcal{G}(\mathbf{1})_{qq}^{(0)} \; ,$$

$$\mathcal{G}(\mathbf{10}+\mathbf{10}^*)_{gg}^{(s)} = -\frac{3}{8}(N_c^2+12) \; , \;\; \mathcal{G}(\mathbf{10}+\mathbf{10}^*)_{gg}^{(v)} = \frac{9}{2}$$

independently on σ;

Four loops

$$\mathcal{G}(\mathbf{8})^{(s)}_{AB} = \mathcal{G}(\mathbf{8})^{(s)c}_{AB} = \mathcal{G}(\mathbf{8})^{(s)d}_{AB} = \mathcal{G}(\mathbf{8})^{(s)e}_{AB} = \mathcal{G}(\mathbf{8})^{(s)}_{AB}\,,$$

$$\mathcal{G}(\mathbf{8})^{(s)b}_{AB} = \mathcal{G}(\mathbf{8})^{(s)c}_{AB} = \mathcal{G}(\mathbf{8})^{(s)d}_{AB} = \mathcal{G}(\mathbf{8})^{(s)e}_{AB} = \mathcal{G}(\mathbf{8})^{(s)}_{AB}\,,$$

$$\frac{1}{2}\left(\mathcal{G}(\mathbf{8})^{(s)a}_{AB} + \mathcal{G}(\mathbf{8})^{(s)f}_{AB}\right) = \mathcal{G}(\mathbf{8})^{(s)}_{AB} + \left(\frac{N_c^4}{16} + \frac{3N_c^2}{8}\right)\,.$$

The important thing which is seen is that the terms violating the pole factorization have σ-independent colour coefficients, which provides their gauge invariance, as well as in the two and three loops.

The colour coefficients are different for different approaches.

It makes possible to discriminate one of them.

The question of whether the four-looop amplitudes of elastic scattering in QCD are given by the Regge pole and cut contributions, in some approach, can be solved by comparing of the corrections with the result obtained using the infrared factorization.

Summary

- The pole Regge form of amplitudes with gluon quantum numbers in cross channels and negative signature, which is the basis of the BFKL equation, is violated in the NNLLA.
- The observed violation can be explained by the cut contributions.
- There are two different explanations of the violation.
- To select the correct form of the cut, a four-loop calculations were performed.
- They have to be compared with results obtained using infrared factorization approach.
- But fixed order calculations can not prove that the QCD amplitudes with gluon quantum numbers in cross-channels and negative signature are given in the NNLLA by the contributions of the Regge pole and the three-Reggeon cut.
- Some idea of the proof, like bootstrap of the gluon Reggeization, but with account of the cut, is required.