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Landau-Khalatnikov-Fradkin transformation and even ζ -values.

OUTLINE

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Abstract

The Landau-Khalatnikov-Fradkin (LKF) transformation is a powerful and elegant transformation allowing to study the gauge dependence of the propagator of charged particles interacting with gauge fields. **With the help of this transformation, we derive a non-perturbative identity between massless propagators in two different gauges. From this identity, we find that the corresponding perturbative series can be exactly expressed in terms of a hatted transcendental basis that eliminates all even ζ -values. Our construction further allows us to derive an exact formula relating hatted and standard ζ -values to all orders of perturbation theory.**

0. Introduction

Gauge invariance governs the dynamics of systems of charged particles with deep consequences in elementary particle physics and beyond. Through the gauge principle, it gives rise to gauge field theories the prototype of which is quantum electrodynamics (QED).

While physical quantities should not depend on this parameter, precious information can be obtained by studying the ξ -dependence of various correlation functions.

Such a task can be carried out with the help of the Landau-Khalatnikov-Fradkin (LKF) transformation (Landau, Khalatnikov: 1956), (Fradkin:1956) that elegantly relates the QED fermion propagator $S_F(p, \xi)$ and $S_F(p, \eta)$ in two different ξ and η gauges. In dimensional regularization, it reads:

$$S_F(x, \xi) = S_F(x, \eta) e^{i(D(x) - D(0))},$$

where

$$D(x) = -i \Delta e^2 \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ipx}}{p^4}, \quad \Delta = \xi - \eta.$$

Let us show basic steps of (Landau, Khalatnikov: 1956).

Gauge invariance arises in the field theory of charged particles interacting with an electromagnetic field. Given a gauge transformation of the potential of electromagnetic field

$$A_\mu \rightarrow A_\mu + \frac{\partial\varphi(x)}{\partial x_\mu},$$

where $\varphi(x)$ is an arbitrary operator function.

The Ψ -function of particle is transformed as follows:

$$\Psi(x) \rightarrow \Psi(x)e^{ie\varphi(x)}$$

Question: how the Green's function $S_F(x)$ for the particles will change under such a gauge transformation.

We would like to note that Fourier components of the Green's function $G_{\mu\nu}(x)$ for photons can be written in the general case in the form

$$G_{\mu\nu}(k) \sim \frac{d_t(k)}{k^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + d_l(k) \frac{k_\mu k_\nu}{k^2},$$

where the terms containing $d_t(k)$ and $d_l(k)$ represent respectively the transverse and longitudinal parts of the function $G_{\mu\nu}$.

Moreover, the longitudinal part does not depend upon interaction with the field.

The Green's function $D_\varphi(x)$ for the $\varphi(x)$ field is connected with the longitudinal part $G_{\mu\nu}^l(x)$ of the Green's function for photons:

$$G_{\mu\nu}^l(x) = \frac{\partial^2 D_\varphi(x)}{\partial x_\mu \partial x_\nu}.$$

So, Fourier components of the Green's function $D_\varphi(x)$ for the $\varphi(x)$ field can be written in the form

$$D_\varphi(k) \sim \frac{d_l(k)}{k^4}$$

with $d_l(k) \sim 1$. **It is very unusual Green's function.**

Taking the above transformation for Ψ -function of particle and using the fact that the operators $\varphi(x)$ represent a free field, Landau, and Khalatnikov found the gauge transformation for the Green's function $S_F(x)$ as

$$S_F(x) = S_F^t(x) \times e^{ie^2(D_\varphi(0) - D_\varphi(x))}$$

where $S_F^t(x)$ is the Green's function in the Landau gauge.

The most important applications of the **Landau-Khalatnikov-Fradkin (LKF) transformation** (Curtis, Pennington: 1990), (Dong, Munczek, Roberts: 1994, 1996), (Bashir, Kizilersu, Pennington: 1998, 2000), (Burden, Tjiang: 1998), (Jia, Pennington: 2016, 2017)

are related to the study of the gauge covariance of QED Schwinger-Dyson equations and their solutions. This allows, *e.g.*, to construct a charged-particle-photon vertex ansatz both in scalar (Fernandez-Rangel, Bashir, Gutierrez-Guerrero, Concha-Sanchez: 2016), (Ahmadiniaz, Bashir, Schubert: 2016) and spinor QED (Kizilersu, Pennington: 2009).

Other applications

(Bashir, Raya: 2002), (Jia, Pennington: 2017) are focused on estimating large orders of perturbation theory. Indeed, the non-perturbative nature of the LKF transformation allows to fix some of the coefficients of the all-order expansion of the fermion propagator. Starting with a perturbative propagator in some fixed gauge, say η , all the coefficients depending on the difference between the gauge fixing parameters of the two propagators, $\xi - \eta$, get fixed by a weak coupling expansion of the LKF-transformed initial one. Such estimations have been carried out for QED in various dimensions (see (Bashir, Raya: 2002), (Jia, Pennington: 2017)), for generalizations to brane worlds (Ahmad, Cobos-Martinez, Concha-Sanchez, Raya: 2016), (James, A.V.K., Teber: 2020) and for more general SU(N) gauge theories (Meerleer, Dudal, Sorella, Dall’Olio, Bashir: 2018).

A seemingly unrelated topic is focused on the multi-loop structure of propagator-type functions (p-functions) [Following \(Baikov, Chetyrkin: 2018\)](#) by p-functions we understand (\overline{MS} -renormalized) Euclidean 2-point functions (that can also be obtained from 3-point functions by setting one external momentum to zero with the help of infra-red rearrangement) expressible in terms of massless propagator-type Feynman integrals also known as p-integrals.

About three decades ago, it was noticed that all contributions proportional to $\zeta_4 = \pi^4/90$ mysteriously cancel out in the Adler function at three-loops ([Gorishnii, Kataev, Larin: 1990](#)). Two decades later, it was shown that the four-loop contribution is also π -free and that a similar fact holds for the coefficient function of the Bjorken sum rule ([Baikov, Chetyrkin, Kühn: 2010](#)).

There is by now mounting evidence, see, *e.g.*, (Baikov, Chetyrkin, Kühn: 2017), (Chetyrkin, Falcioni, Herzog, Vermaseren: 2017), (Herzog, Ruijl, Ueda, Vermaseren, Vogt: 2017, 2018), (Davies, Vogt: 2018), (Moch, Ruijl, Ueda, Vermaseren, Vogt: 2018), (Vogt, Herzog, Moch, Ruijl, Ueda, Vermaseren: 2018), that various massless Euclidean physical quantities demonstrate striking regularities in terms proportional to even ζ -function values, ζ_{2n} , *e.g.*, to π^{2n} with n being a positive integer.

Additional cancellations of π^{2n} terms have been observed in the so-called C-scheme (Jamin and R. Miravitllas: 2018).

Such puzzling facts have recently given rise to the “no- π theorem”. The latter is based on the observation (Broadhurst: 1999), (Baikov, Chetyrkin: 2010, 2018) that the ε -dependent transformation of the ζ -values:

$$\hat{\zeta}_3 \equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \quad \hat{\zeta}_5 \equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \quad \hat{\zeta}_7 \equiv \zeta_7,$$

eliminates even zetas from the expansion of four-loop p-integrals. A generalization to 5-, 6- and 7-loops is available in (Georgoudis, Goncalves, Panzer, Pereira: 2018), (Baikov, Chetyrkin: 2018, 2019), respectively.

!!! Note that these results also contain multi-zeta values the consideration of which is beyond the scope of the present study. !!!

Definition:

$$\zeta_a = \sum_{k>1} \frac{1}{k^a}, \quad \zeta_{a,b} = \sum_{k>m>1} \frac{1}{k^a m^b}, \quad \zeta_{a,b,c} = \sum_{k>m>l>1} \frac{1}{k^a m^b l^c}.$$

In the present paper, we shall use the LKF transformation in order to study general properties of the coefficients of the propagator. We will show how the transformation naturally reveals the existence of the hatted transcendental basis. Moreover, it will allow us to extend the above results to any order in ε .

The appearance of the hatted transcendental basis from the LKF transformation can be naturally understood in the following way.

The LKF transformation produces all-loop results for very restricted objects: the difference of fermion propagators in two gauges. So, at every order of the ε -expansion these all-loop results should contain (at least, a part of) the basic properties of the corresponding master integrals, *i.e.*, the all-loop results should be expressed in the form of (at least, a part of) the corresponding hatted ζ -values.

In a sense, it is not the full set of the hatted ζ -values but only the one-fold ones. This comes from the fact that the results produced by the LKF transformation contain only products of Γ -functions and, thus, their expansions contain only the simple one-fold ζ -values.

1. LKF transformation

In the following, we shall consider QED in an Euclidean space of dimension d ($d = 4 - 2\varepsilon$). The general form of the fermion propagator $S_F(p, \xi)$ in some gauge ξ reads:

$$S_F(p, \xi) = \frac{1}{i\hat{p}} P(p, \xi),$$

where the factor \hat{p} containing Dirac γ -matrices, has been extracted. It is also convenient to introduce the x -space representation $S_F(x, \xi)$ of the fermion propagator as:

$$S_F(x, \xi) = \hat{x} X(x, \xi).$$

The two representations, $S_F(x, \xi)$ and $S_F(p, \xi)$, are related by the Fourier transform which is defined as:

$$S_F(p, \xi) = \int \frac{d^d x}{(2\pi)^{d/2}} e^{ipx} S_F(x, \xi),$$

$$S_F(x, \xi) = \int \frac{d^d p}{(2\pi)^{d/2}} e^{-ipx} S_F(p, \xi).$$

The famous LKF transformation connects in a very simple way the fermion propagator in two different gauges, *e.g.*, ξ and η . In dimensional regularization, it reads:

$$S_F(x, \xi) = S_F(x, \eta) e^{i(D(x) - D(0))},$$

where

$$D(x) = -i \Delta e^2 \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ipx}}{p^4}, \quad \Delta = \xi - \eta.$$

Note that, in dimensional regularization, the term $D(0)$ is proportional to the massless tadpole T_2 , the massive counterpart of which is defined as:

$$T_\alpha(m^2) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ipx}}{(p^2 + m^2)^\alpha}.$$

The tadpole $T_\alpha(m^2) \sim \delta(\alpha - d/2)$ in the massless limit and, thus, $D(0) = 0$ in the framework of dimensional regularization. So, the LKF transformation can be simplified as follows:

$$S_F(x, \xi) = S_F(x, \eta) e^{iD(x)}.$$

We may now proceed in calculating $D(x)$ using the Fourier transforms

$$\int d^d x \frac{e^{ipx}}{x^{2\alpha}} = \frac{2^{2\tilde{\alpha}} \pi^{d/2} a(\alpha)}{p^{2\tilde{\alpha}}}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = \frac{d}{2} - \alpha,$$

$$\int d^d p \frac{e^{-ipx}}{p^{2\alpha}} = \frac{2^{2\tilde{\alpha}} \pi^{d/2} a(\alpha)}{x^{2\tilde{\alpha}}}.$$

This yields:

$$D(x) = -i \Delta e^2 (\mu^2 x^2)^{2-d/2} \frac{\Gamma(d/2 - 2)}{2^4 (\pi)^{d/2}},$$

or, equivalently, with the parameter ε made explicit:

$$D(x) = \frac{i \Delta A}{\varepsilon} \Gamma(1 - \varepsilon) (\pi \mu^2 x^2)^\varepsilon, \quad A = \frac{e^2}{(4\pi)^2}.$$

We see that $D(x)$ contributes with a common factor ΔA accompanied by the singularity ε^{-1} .

Note that, in the case of the scalar QED, the corresponding scalar propagator $S_C(x, \xi)$ has a similar transformation between ξ and η gauges, *i.e.*,

$$S_C(x, \xi) = S_C(x, \eta) e^{iD(x)} .$$

2. LKF transformation in momentum space

Let's assume that, for some gauge fixing parameter η , the fermion propagator $S_F(p, \eta)$ with external momentum p has the form

$$S_F(p, \xi) = \frac{1}{i\hat{p}} P(p, \xi), \quad P(p, \eta) = \sum_{m=0}^{\infty} a_m(\eta) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon}.$$

The $a_m(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ is the renormalization scale:

$$\tilde{\mu}^2 = 4\pi\mu^2,$$

which lies somehow between the MS-scale μ and the $\overline{\text{MS}}$ -scale $\bar{\mu}$.

Then, the LKF transformation shows that, for another gauge parameter ξ , the fermion propagator can be expressed as:

$$P(p, \xi) = \sum_{m=0}^{\infty} a_m(\xi) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon},$$

where

$$a_m(\xi) = a_m(\eta) \frac{\Gamma(2 - (m + 1)\varepsilon)}{\Gamma(1 + m\varepsilon)} \sum_{l=0}^{\infty} \frac{\Gamma(1 + (m + l)\varepsilon) \Gamma^l(1 - \varepsilon) (\Delta A)^l}{l! \Gamma(2 - (m + l + 1)\varepsilon)} \frac{1}{(-\varepsilon)^l} \left(\frac{\tilde{\mu}^2}{p^2} \right)^{l\varepsilon}$$

In order to derive it, we used the fermion propagator $S_F(p, \eta)$ with $P(p, \eta)$, did the Fourier transform to $S_F(x, \eta)$ and applied the LKF transformation. As a final step, we took the inverse Fourier transform and obtained $S_F(p, \xi)$ with $P(p, \xi)$.

2.1 Scale fixing

In our present study, we consider only the case of the so-called MS-like schemes. In such schemes, we need to fix specific terms coming from the application of dimensional regularization. Such a procedure will be called *scale fixing* and will play a crucial role in our analysis.

Let's first recall that the $\overline{\text{MS}}$ -scale $\bar{\mu}$ is related to the previously defined scale $\tilde{\mu}$ with the help of:

$$\begin{aligned}\bar{\mu}^2 &= \tilde{\mu}^2 e^{-\gamma} \text{(simplest possibility),} \\ [\bar{\mu}^{2\varepsilon} &= \tilde{\mu}^{2\varepsilon} \Gamma(1 + \varepsilon), \quad \bar{\mu}^{2l\varepsilon} = \tilde{\mu}^{2l\varepsilon} \Gamma(1 + l\varepsilon), \text{(other possibilities)}]\end{aligned}$$

where γ is the Euler constant. An advantage of the $\overline{\text{MS}}$ -scale is that it subtracts the Euler constant γ from the ε -expansion.

Moreover, it is well known that, in calculations of two-point massless diagrams, the final results do not display any ζ_2 . So it is convenient to choose some scale which also subtracts ζ_2 in intermediate steps of the calculation. For this purpose, we shall consider two different scales.

The first one is the popular G -scale (Chetyrkin, Kataev, Tkachov: 1980),

which subtracts the coefficient in factor of the singularity $1/\varepsilon$ in the one-loop scalar p-type integral, *i.e.*,

$$\mu_G^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \frac{\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)}{\Gamma(2 - 2\varepsilon)}.$$

Following (Broadhurst: 1999),

we shall use a slight modification of this scale that we will refer to as the g -scale and in which an additional factor $1/(1 - 2\varepsilon)$ is subtracted from the one-loop result, *i.e.*,

$$\mu_g^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \frac{\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)}{\Gamma(1 - 2\varepsilon)}.$$

The advantage of the g -scale (over the G -scale) will reveal itself in discussions below related to the so-called transcendental weight of various contributions.

We shall also introduce a new scale which is based on old calculations of massless diagrams performed by Vladimirov (Vladimirov: 1980), who added an additional factor $\Gamma(1-\varepsilon)$ to each loop contribution. The latter corresponds to adding the factor $\Gamma^{-1}(1-\varepsilon)$ to the corresponding scale. We shall refer to this scale as the minimal Vladimirov-scale, or MV-scale, and define

$$\mu_{\text{MV}}^{2\varepsilon} = \frac{\tilde{\mu}^{2\varepsilon}}{\Gamma(1-\varepsilon)}.$$

Notice that this form has been used once to define the $\overline{\text{MS}}$ scheme (see Errata to (Kataev, Vardiashvili: 1988).)

As we will show below, the use of the MV-scale leads to simpler results in comparison with the g one. Hence, the MV-scale is more appropriate to our analysis and all our basic results will be given in the MV-scale. After that we will discuss the differences coming from the use of the g -scale.

In both the MV-scale and g -scale, we can rewrite the above result in the following general form:

$$a_m(\xi) = a_m(\eta) \sum_{l=0}^{\infty} \frac{1 - (m+1)\varepsilon}{1 - (m+l+1)\varepsilon} \Phi_p(m, l, \varepsilon) \frac{(\Delta A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_p^2}{p^2} \right)^{l\varepsilon},$$

where $p = \text{MV}, g$.

The factor $(1 - (m+1)\varepsilon)/(1 - (m+l+1)\varepsilon)$ has been specially extracted from $\Phi_p(m, l, \varepsilon)$ in order to insure equal transcendental level, *i.e.*, the same value of s for ζ_s at every order of the ε -expansion of $\Phi_p(m, l, \varepsilon)$ (see below).

Central to the present work, the factors $\Phi_{\text{MV}}(m, l, \varepsilon)$ and $\Phi_g(m, l, \varepsilon)$ read:

$$\Phi_{\text{MV}}(m, l, \varepsilon) = \frac{\Gamma(1 - (m + 1)\varepsilon)\Gamma(1 + (m + l)\varepsilon)\Gamma^{2l}(1 - \varepsilon)}{\Gamma(1 + m\varepsilon)\Gamma(1 - (m + l + 1)\varepsilon)},$$

$$\Phi_g(m, l, \varepsilon) = \Phi_{\text{MV}}(m, l, \varepsilon) \frac{\Gamma^l(1 - 2\varepsilon)}{\Gamma^{3l}(1 - \varepsilon)\Gamma^l(1 + \varepsilon)},$$

and may be expressed as expansions in ζ_i ($i \geq 3$).

2.2 Scalar QED

In the case of scalar (spin-0) QED, the LKF transformation leads to expressions which are very similar to spinor QED that we have considered so far.

Indeed, let $S_C(p, \eta)$ be a scalar propagator with external momentum p and gauge fixing parameter η ; its general expression reads:

$$S_C(p, \eta) = \frac{1}{p^2} \sum_{m=0}^{\infty} a_m^c(\eta) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon},$$

where $a_m^c(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ the renormalization scale.

Then, the LKF transformation shows that, for another gauge parameter ξ , the scalar propagator can be expressed as:

$$S_C(p, \xi) = \frac{1}{p^2} \sum_{m=0}^{\infty} a_m^c(\xi) A^m \left(\frac{\tilde{\mu}^2}{p^2} \right)^{m\varepsilon},$$

where

$$a_m^c(\xi) = a_m^c(\eta) \frac{\Gamma(1 - (m + 1)\varepsilon)}{\Gamma(1 + m\varepsilon)} \sum_{l=0}^{\infty} \frac{\Gamma(1 + (m + l)\varepsilon) \Gamma^l(1 - \varepsilon) (\Delta A)^l}{l! \Gamma(1 - (m + l + 1)\varepsilon)} \frac{1}{(-\varepsilon)^l} \left(\frac{\tilde{\mu}^2}{p^2} \right)^{l\varepsilon}$$

where we used the scalar propagator $S_C(p, \eta)$, did the Fourier transform to $S_C(x, \eta)$ and applied the LKF transformation. As a final step, we took the inverse Fourier transform and obtained $S_C(p, \xi)$.

For the MV- and g -scales, we have

$$a_m^c(\xi) = a_m^c(\eta) \sum_{l=0}^{\infty} \Phi_p(m, l, \varepsilon) \frac{(\Delta A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_p^2}{p^2} \right)^{l\varepsilon},$$

where $p = \text{MV}, g$. We therefore see that the difference between the spinor and scalar cases is only in the factor

$$(1 - (m + 1)\varepsilon) / (1 - (m + l + 1)\varepsilon)$$

which did not play any role in the above analysis. Hence, for scalar QED, we can repeat all evaluations done for spinor QED and we will end up with the same set of hatted ζ -values.

3. MV-scale

The Γ -function $\Gamma(1 + \beta\varepsilon)$ has the following expansion:

$$\Gamma(1 + \beta\varepsilon) = \exp \left[-\gamma\beta\varepsilon + \sum_{s=2}^{\infty} (-1)^s \eta_s \beta^s \varepsilon^s \right], \quad \eta_s = \frac{\zeta_s}{s}.$$

that yields for the factor $\Phi_{\text{MV}}(m, l, \varepsilon)$:

$$\Phi_{\text{MV}}(m, l, \varepsilon) = \exp \left[\sum_{s=2}^{\infty} \eta_s p_s(m, l) \varepsilon^s \right],$$

where

$$p_s(m, l) = (m+1)^s - (m+l+1)^s + 2l + (-1)^s \{(m+l)^s - m^s\},$$

and, as expected from the MV-scale, we do have:

$$p_1(m, l) = 0, \quad p_2(m, l) = 0.$$

Moreover $\Phi_{\text{MV}}(m, l, \varepsilon)$ contains ζ_s -function values of a given weight (or transcendental level) s in factor of ε^s .

4. Solution of the recurrence relations

We now focus on the polynomial $p_s(m, l)$ that is conveniently separated in even and odd s values. Then, we see that the following recursion relations hold:

$$p_{2k} = p_{2k-1} + Lp_{2k-2} + p_3, \quad L = l(l + 1),$$

$$p_{2k-1} = p_{2k-2} + Lp_{2k-3} + p_3.$$

Specific to the MV-scheme, these relations only depend on L which leads to strong simplifications.

Nevertheless, they are difficult to solve for arbitrary k . It is simpler to proceed by explicitly considering the first values of k :

$$p_4 = 2p_3,$$

$$p_5 = p_4 + Lp_3 + p_3 = (3 + L)p_3,$$

$$p_6 = p_5 + Lp_4 + p_3 = (4 + 3L)p_3,$$

showing that p_s takes the form of a polynomial in L in factor of p_3 . Then, taking Lp_3 from the second equation and put it to the thirs one, yields:

$$Lp_3 = p_5 - 3p_3, \quad p_6 = 3p_5 - 5p_3,$$

which reveals that the even polynomial p_6 can be entirely expressed in terms of the lower order odd ones, p_3 and p_5 .

We may automate this procedure for higher values of k . The general expression of p_s is given by:

$$p_s = \sum_{m=0}^{\lfloor \frac{s+1}{2} - 2 \rfloor} A_{s,m} L^m p_3.$$

Taking $L^k p_3$ from the equations for p_{2k-1} and substituting them in the equations for p_{2k} yields:

$$p_{2k} = \sum_{s=2}^k p_{2s-1} C_{2k,2s-1} = \sum_{m=1}^{k-1} p_{2k-2m+1} C_{2k,2k-2m+1}.$$

From these results, it is possible to determine the exact k -dependence of $C_{2k,2s-1}$, which has the following structure:

$$C_{2k,2k-2m+1} = b_{2m-1} \frac{(2k)!}{(2m-1)!(2k-2m+1)!},$$

with the first coefficients b_{2m-1} taking the values:

$$\begin{aligned} b_1 &= \frac{1}{2}, & b_3 &= -\frac{1}{4}, & b_5 &= \frac{1}{2}, & b_7 &= -\frac{17}{2}, & b_9 &= \frac{31}{2}, \\ b_{11} &= -\frac{691}{4}, & b_{13} &= \frac{5461}{2}, & b_{15} &= -\frac{929569}{16}, \\ b_{17} &= \frac{3202291}{2}, & b_{19} &= -\frac{221930581}{4}, \\ b_{21} &= \frac{4722116521}{2}, & b_{23} &= -\frac{968383680827}{8}. \end{aligned}$$

Examining the numerators of b_{2m-1} , one can see that they are proportional to the numerators of Bernoulli numbers. Indeed, a closer inspection reveals that, accurate to a sign, the coefficients

b_{2m-1} coincide with the zero values of Euler polynomials $E_n(x)$:

$$b_{2m-1} = -E_{2m-1}(x=0),$$

and therefore to Bernoulli and Genocchi numbers, B_m and G_m , respectively, because

$$E_{2m-1}(x=0) = \frac{G_{2m}}{2m}, \quad G_{2m} = -2(2^{2m} - 1)B_{2m}.$$

Hence, the compact formula for the coefficients b_{2m-1} , expressed through the well known Bernoulli numbers B_m , reads:

$$b_{2m-1} = \frac{(2^{2m} - 1)}{m} B_{2m}.$$

4.1 Hatted ζ -values

At this point, it is convenient to represent the argument of the exponential as follows:

$$\sum_{s=3}^{\infty} \eta_s p_s \varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1} p_{2k-1} \varepsilon^{2k-1} .$$

Then

$$\begin{aligned} \sum_{k=2}^{\infty} \eta_{2k} p_{2k} \varepsilon^{2k} &= \sum_{k=2}^{\infty} \eta_{2k} \varepsilon^{2k} \sum_{s=2}^k p_{2s-1} C_{2k,2s-1} \\ &= \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k} C_{2k,2s-1} \varepsilon^{2k} . \end{aligned}$$

Then, can be written as $\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} p_{2s-1} \varepsilon^{2s-1}$ where

$$\hat{\eta}_{2s-1} = \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k} C_{2k,2s-1} \varepsilon^{2(k-s)+1} .$$

Thus, we have

$$\Phi_{\text{MV}}(m, l, \varepsilon) = \exp \left[\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} p_{2s-1} \varepsilon^{2s-1} \right] = \exp \left[\sum_{s=2}^{\infty} \frac{\hat{\zeta}_{2s-1}}{2s-1} p_{2s-1} \varepsilon^{2s-1} \right],$$

where

$$\hat{\zeta}_{2s-1} = \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k} \hat{C}_{2k, 2s-1} \varepsilon^{2(k-s)+1}$$

with

$$C_{2k, 2s-1} = b_{2k-2s+1} \frac{(2k)!}{(2s-1)!(2k-2s+1)!},$$

$$\hat{C}_{2k, 2s-1} = \frac{2s-1}{2k} C_{2k, 2s-1} = b_{2k-2s+1} \frac{(2k-1)!}{(2s-2)!(2k-2s+1)!}.$$

So, we provide an exact expression for the hatted ζ -values in terms of the standard ones valid for all ε .

4.2 g -scale

We may proceed in a similar way for the factor $\Phi_g(m, l, \varepsilon)$, which has the form

$$\Phi_g(m, l, \varepsilon) = \exp \left[\sum_{s=2}^{\infty} \eta_s p_s^g(m, l) \varepsilon^s \right],$$

where the new polynomial $p_s^g(m, l)$ can be expressed in terms of $p_s(m, l)$, as:

$$p_s^g(m, l) = p_s(m, l) + \delta_s(m, l), \quad \delta_s(m, l) = (2^s - 3 - (-1)^s)l,$$

where $\delta_s(m, l) = 0$ for $s = 1$ and $s = 2$ and, thus,

$$p_1^g(m, l) = 0, \quad p_2^g(m, l) = 0, \quad (1)$$

similarly to the Vladimirov case, considered earlier.

We may then consider the even and odd values of s separately leading to the following recursion relations:

$$p_{2k}^g = p_{2k} + \delta_{2k}, \quad \delta_{2k} = 4(2^{2k-2} - 1)l,$$

$$p_{2k-1}^g = p_{2k-1} + \delta_{2k-1}, \quad \delta_{2k-1} = \frac{1}{2}\delta_{2k}.$$

These recurrence relations depend on the variable l but not on the product $L = l(l+1)$ as it was for the MV-scale. So, the g -scale recursion relations are essentially more complicated than the MV-scale ones. Fortunately, it is very simple to see that in the relations:

$$p_{2k}^g = \sum_{s=2}^k p_{2s-1}^g C_{2k,2s-1},$$

the coefficients $C_{2k,2s-1}$ are exactly the same as earlier because the corrections δ_{2k} and δ_{2k-1} exactly cancel each other. So, the hatted ζ -values for the g -scale are identical to the ones of the MV-scale.

5. Summary

From the LKF transformation of the fermion and scalar propagators we have found peculiar recursion relations between even and odd values of the polynomial associated to the uniformly transcendental factor $\Phi_{\text{MV}}(m, l, \varepsilon)$.

These relations are simple in the new MV-scheme. They relate the even and odd parts in a rather simple way which reveals the possibility to express all results for $\Phi_{\text{MV}}(m, l, \varepsilon)$ in terms of hatted ζ -values.

In the more popular g -scheme, the corresponding recursion relations are slightly more complicated but lead to the same relations between even and odd parts of the polynomial associated to $\Phi_g(m, l, \varepsilon)$ and, correspondingly, to the same hatted ζ -values.

Our careful study of the recursion relations allowed us to derive exact formulas, relating hatted and standard ζ -values to all orders of perturbation theory.

The coefficients of the relations are expressed through the well-known Bernoulli numbers, B_{2m} .

Our results provide stringent constraints on multi-loop calculations at any order in perturbation theory.

What about the multi-zeta values? It is open question!

A. Explicit expressions for $A_{k,m}$, p_s , $C_{2k,2s-1}$ and hatted ζ -values

A.1 Coefficients $A_{k,m}$ for $k \leq 20$

$$A_{k,0} = k - 2, \quad A_{2k-1,k-2} = 1, \quad A_{2k,k-2} = k,$$

$$A_{7,1} = 6, \quad A_{8,1} = 10, \quad A_{9,1} = 15, \quad A_{9,2} = 10, \quad A_{10,1} = 21, \quad A_{10,2} = 20,$$

$$A_{11,1} = 28, \quad A_{11,2} = 35, \quad A_{11,3} = 15,$$

$$A_{12,1} = 36, \quad A_{12,2} = 56, \quad A_{12,3} = 35,$$

$$A_{13,1} = 45, \quad A_{13,2} = 84, \quad A_{13,3} = 70, \quad A_{13,4} = 21,$$

$$A_{14,1} = 55, \quad A_{14,2} = 120, \quad A_{14,3} = 126, \quad A_{14,4} = 56,$$

$$A_{15,1} = 66, \quad A_{15,2} = 165, \quad A_{15,3} = 210, \quad A_{15,4} = 126, \quad A_{15,5} = 28,$$

$$A_{16,1} = 78, \quad A_{16,2} = 220, \quad A_{16,3} = 330, \quad A_{16,4} = 252, \quad A_{16,5} = 84,$$

$$A_{17,1} = 91, \quad A_{17,2} = 286, \quad A_{17,3} = 495, \quad A_{17,4} = 462, \quad A_{17,5} = 210,$$

$$A_{17,6} = 36,$$

$$A_{18,1} = 105, \quad A_{18,2} = 364, \quad A_{18,3} = 715, \quad A_{18,4} = 792, \quad A_{18,5} = 462,$$

$$A_{18,6} = 120,$$

$$A_{19,1} = 120, \quad A_{19,2} = 455, \quad A_{19,3} = 1001, \quad A_{19,4} = 1287,$$

$$A_{19,5} = 924, \quad A_{19,6} = 330, \quad A_{19,7} = 45,$$

$$A_{20,1} = 136, \quad A_{20,2} = 560, \quad A_{20,3} = 1365, \quad A_{20,4} = 2002,$$

$$A_{20,5} = 1716, \quad A_{20,6} = 792, \quad A_{20,7} = 165.$$

A.2 Polynomials p_s for $7 \leq s \leq 20$

$$p_7 = (5 + 6L + L^2)p_3,$$

$$p_8 = (6 + 10L + 4L^2)p_3,$$

$$p_9 = (7 + 15L + 10L^2 + L^3)p_3,$$

$$p_{10} = (8 + 21L + 20L^2 + 5L^3)p_3,$$

$$p_{11} = (9 + 28L + 35L^2 + 15L^3 + L^4)p_3,$$

$$p_{12} = (10 + 36L + 56L^2 + 35L^3 + 6L^4)p_3,$$

$$p_{13} = (11 + 45L + 84L^2 + 70L^3 + 21L^4 + L^5)p_3,$$

$$p_{14} = (12 + 55L + 120L^2 + 126L^3 + 56L^4 + 7L^5)p_3,$$

$$p_{15} = (13 + 66L + 165L^2 + 210L^3 + 126L^4 + 28L^5 + L^6)p_3,$$

$$p_{16} = (14 + 78L + 220L^2 + 330L^3 + 252L^4 + 84L^5 + 8L^6)p_3,$$

$$p_{17} = (15 + 91L + 286L^2 + 495L^3 + 462L^4 + 210L^5 + 36L^6 + L^7)p_3,$$

$$\begin{aligned}
p_{18} &= (16 + 105L + 364L^2 + 715L^3 + 792L^4 + 462L^5 + 120L^6 + 9L^7)p_3, \\
p_{19} &= (17 + 120L + 455L^2 + 1001L^3 + 1287L^4 + 924L^5 + 330L^6 \\
&\quad + 45L^7 + L^8)p_3, \\
p_{20} &= (18 + 136L + 560L^2 + 1365L^3 + 2002L^4 + 1716L^5 + 792L^6 \\
&\quad + 165L^7 + 10L^8)p_3.
\end{aligned}$$

A.3 Coefficients $C_{2k,2s-1}$ for $k \leq 10$

$$C_{2k,2k-1} = k, \quad C_{6,3} = -5, \quad C_{8,5} = -14, \quad C_{8,3} = 28,$$

$$C_{10,7} = -30, \quad C_{10,5} = 126, \quad C_{10,3} = -255,$$

$$C_{12,9} = -55, \quad C_{12,7} = 396, \quad C_{12,5} = -1683, \quad C_{12,3} = 3410,$$

$$C_{14,11} = -91, \quad C_{14,9} = 1001, \quad C_{14,7} = -7293, \quad C_{14,5} = 31031,$$

$$C_{14,3} = -62881,$$

$$C_{16,13} = -140, \quad C_{16,11} = 2184, \quad C_{16,9} = -24310, \quad C_{16,7} = 177320,$$

$$C_{16,5} = -754572, \quad C_{16,3} = 1529080,$$

$$C_{18,15} = -204, \quad C_{18,13} = 4284, \quad C_{18,11} = -67626, \quad C_{18,9} = 753610,$$

$$C_{18,7} = -5497596, \quad C_{18,5} = 23394924, \quad C_{18,3} = -47408019,$$

$$C_{20,17} = -285, \quad C_{20,15} = 7752, \quad C_{20,13} = -164730, \quad C_{20,11} = 2603380,$$

$$C_{20,9} = -29015090, \quad C_{20,7} = 211668360, \quad C_{20,5} = -900752361,$$

$$C_{20,3} = 1825305870.$$

A.4 Hatted ζ -values, $\hat{\zeta}_{2s-1}$, for $s \leq 10$

We display below the first terms of the ε -series generated by our exact result. For the sake of clarity, we display in blue the terms which were known up to the present work from the latest publication (Baikov, Chetyrkin: 2018) (and in red from the new paper (Baikov, Chetyrkin: 2019) published after our the present paper)

$$\begin{aligned} \hat{\zeta}_3 &= \zeta_3 + \frac{3\varepsilon}{2} \zeta_4 - \frac{5\varepsilon^3}{2} \zeta_6 + \frac{21\varepsilon^5}{2} \zeta_8 - \frac{153\varepsilon^7}{2} \zeta_{10} + \frac{1705\varepsilon^9}{2} \zeta_{12} \\ &\quad - \frac{26949\varepsilon^{11}}{2} \zeta_{14} + \frac{573405\varepsilon^{13}}{2} \zeta_{16} - \frac{15802673\varepsilon^{15}}{2} \zeta_{18} + \\ &\quad + \frac{547591761\varepsilon^{17}}{2} \zeta_{20} - \frac{23302711005\varepsilon^{19}}{2} \zeta_{22} + \frac{1194695479813\varepsilon^{21}}{2} \zeta_{24} \\ &\quad - \frac{72628776062025\varepsilon^{23}}{2} \zeta_{26} + \frac{5165901157067001\varepsilon^{25}}{2} \zeta_{28} \end{aligned}$$

$$-\frac{425013158488292213 \varepsilon^{27}}{2} \zeta_{30} + \frac{40048437741888549165 \varepsilon^{29}}{2} \zeta_{32} + O(\varepsilon^{31}),$$

$$\begin{aligned} \hat{\zeta}_5 &= \zeta_5 + \frac{5\varepsilon}{2} \zeta_6 - \frac{35\varepsilon^3}{4} \zeta_8 + 63\varepsilon^5 \zeta_{10} - \frac{2805\varepsilon^7}{4} \zeta_{12} + \frac{22165\varepsilon^9}{2} \zeta_{14} \\ &- \frac{943215\varepsilon^{11}}{4} \zeta_{16} + 6498590\varepsilon^{13} \zeta_{18} - \frac{900752361\varepsilon^{15}}{4} \zeta_{20} \\ &+ \frac{19165711635\varepsilon^{17}}{2} \zeta_{22} - \frac{1965195294755\varepsilon^{19}}{4} \zeta_{24} \\ &+ 29867386995325\varepsilon^{21} \zeta_{26} - \frac{8497566799256925\varepsilon^{23}}{4} \zeta_{28} \\ &+ \frac{349559311628200401\varepsilon^{25}}{2} \zeta_{30} - \frac{65877039565685293015\varepsilon^{27}}{4} \zeta_{32} + O(\varepsilon^{29}), \end{aligned}$$

$$\begin{aligned} \hat{\zeta}_7 &= \zeta_7 + \frac{7\varepsilon}{2} \zeta_8 - 21\varepsilon^3 \zeta_{10} + 231\varepsilon^5 \zeta_{12} - \frac{7293\varepsilon^7}{2} \zeta_{14} + \frac{155155\varepsilon^9}{2} \zeta_{16} \\ &- 2137954\varepsilon^{11} \zeta_{18} + 74083926\varepsilon^{13} \zeta_{20} - \frac{6305266527\varepsilon^{15}}{2} \zeta_{22} \end{aligned}$$

$$\begin{aligned}
& + \frac{323261669577 \varepsilon^{17}}{2} \zeta_{24} - 9825976473775 \varepsilon^{19} \zeta_{26} \\
& + 698896855690605 \varepsilon^{21} \zeta_{28} - \frac{115000404016610385 \varepsilon^{23}}{2} \zeta_{30} \\
& + \frac{10836338660474212431 \varepsilon^{25}}{2} \zeta_{32} - 579717948178030578532 \varepsilon^{27} \zeta_{34} + O(\varepsilon^{29})
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_9 & = \zeta_9 + \frac{9\varepsilon}{2} \zeta_{10} - \frac{165\varepsilon^3}{4} \zeta_{12} + \frac{1287\varepsilon^5}{2} \zeta_{14} - \frac{109395\varepsilon^7}{8} \zeta_{16} \\
& + 376805 \varepsilon^9 \zeta_{18} - \frac{26113581 \varepsilon^{11}}{2} \zeta_{20} + 555629445 \varepsilon^{13} \zeta_{22} \\
& - \frac{227890347333 \varepsilon^{15}}{8} \zeta_{24} + \frac{3463517888325 \varepsilon^{17}}{2} \zeta_{26} \\
& - \frac{492702534613575 \varepsilon^{19}}{4} \zeta_{28} + \frac{20268008815027545 \varepsilon^{21}}{2} \zeta_{30} \\
& - \frac{7639312552531975575 \varepsilon^{23}}{8} \zeta_{32} + 102171193084471145778 \varepsilon^{25} \zeta_{34} \\
& - 12319006398783149793805 \varepsilon^{27} \zeta_{36} + O(\varepsilon^{29}),
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_{11} &= \zeta_{11} + \frac{11\varepsilon}{2}\zeta_{12} - \frac{143\varepsilon^3}{2}\zeta_{14} + \frac{3003\varepsilon^5}{2}\zeta_{16} - 41327\varepsilon^7\zeta_{18} \\
&+ 1431859\varepsilon^9\zeta_{20} - 60931689\varepsilon^{11}\zeta_{22} + 3123872213\varepsilon^{13}\zeta_{24} - \\
&\frac{37981724555\varepsilon^{15}}{2}\zeta_{26} + \frac{27015439528935\varepsilon^{17}}{2}\zeta_{28} \\
&\frac{2222635878367905\varepsilon^{19}}{2}\zeta_{30} + \frac{209436091088617965\varepsilon^{21}}{2}\zeta_{32} \\
&- 11204325077046897510\varepsilon^{23}\zeta_{34} + 1350930219672451816398\varepsilon^{25}\zeta_{36} + O(\varepsilon)
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_{13} &= \zeta_{13} + \frac{13\varepsilon}{2}\zeta_{14} - \frac{455\varepsilon^3}{4}\zeta_{16} + 3094\varepsilon^5\zeta_{18} - \frac{214149\varepsilon^7}{2}\zeta_{20} \\
&+ 4555915\varepsilon^9\zeta_{22} - \frac{467142949\varepsilon^{11}}{2}\zeta_{24} + 14199419150\varepsilon^{13}\zeta_{26} \\
&\frac{4039874339085\varepsilon^{15}}{4}\zeta_{28} + \frac{166185885587085\varepsilon^{17}}{2}\zeta_{30} \\
&\frac{31318960104275025\varepsilon^{19}}{4}\zeta_{32} + 837744364354471860\varepsilon^{21}\zeta_{34} \\
&- 101008688194589454825\varepsilon^{23}\zeta_{36} + O(\varepsilon^{25}),
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_{15} &= \zeta_{15} + \frac{15 \varepsilon}{2} \zeta_{16} - 170 \varepsilon^3 \zeta_{18} + 5814 \varepsilon^5 \zeta_{20} - 247095 \varepsilon^7 \zeta_{22} \\
&+ 12666445 \varepsilon^9 \zeta_{24} - 770015850 \varepsilon^{11} \zeta_{26} + 54769188150 \varepsilon^{13} \zeta_{28} \\
&- \frac{9012027371805 \varepsilon^{15}}{2} \zeta_{30} + \frac{849191613164775 \varepsilon^{17}}{2} \zeta_{32} \\
&- 45429700371036300 \varepsilon^{19} \zeta_{34} + 5477559305394623700 \varepsilon^{21} \zeta_{36} + O(\varepsilon^{23}), \\
\hat{\zeta}_{17} &= \zeta_{17} + \frac{17 \varepsilon}{2} \zeta_{18} - \frac{969 \varepsilon^3}{4} \zeta_{20} + \frac{20349 \varepsilon^5}{2} \zeta_{22} - \frac{4167669 \varepsilon^7}{8} \zeta_{24} \\
&+ \frac{63332225 \varepsilon^9}{2} \zeta_{26} - \frac{9009185445 \varepsilon^{11}}{4} \zeta_{28} + \frac{370604839815 \varepsilon^{13}}{2} \zeta_{30} \\
&- \frac{279372848525955 \varepsilon^{15}}{16} \zeta_{32} + 1868221548962505 \varepsilon^{17} \zeta_{34} \\
&- \frac{450511195346109975 \varepsilon^{19}}{2} \zeta_{36} + \\
&+ O(\varepsilon^{21}), \\
\hat{\zeta}_{19} &= \zeta_{19} + \frac{19 \varepsilon}{2} \zeta_{20} - \frac{665 \varepsilon^3}{2} \zeta_{22} + \frac{33649 \varepsilon^5}{2} \zeta_{24} - \frac{2042975 \varepsilon^7}{2} \zeta_{26}
\end{aligned}$$

$$\begin{aligned}
& + \frac{145291575 \varepsilon^9}{2} \zeta_{28} - \frac{11953363695 \varepsilon^{11}}{4} \zeta_{30} + \frac{1126348042575 \varepsilon^{13}}{2} \zeta_{32} \\
& - 6025688897755 \varepsilon^{15} \zeta_{34} + 7265306023743075 \varepsilon^{17} \zeta_{36} + O(\varepsilon^{19}).
\end{aligned}$$