Matching the heavy-quark fields in QCD and HQET at four loops

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QCD and HQET

- QCD: single heavy quark Q with $P = Mv + k$, $k \ll M$
- HQET: h_v with k

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Matching on-shell matrix elements

$$<0|Q_0(x)|Q(P)> = e^{-iP\cdot x}\left(Z_Q^{\mathrm{os}}\right)^{1/2} u(P)$$

$$<0|h_{v0}(x)|h(k)> = e^{-ik\cdot x}\,(Z_h^{\mathrm{os}})^{1/2}\, u_v(P)$$

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- HQET: h_v with k

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$$<0|h_{v0}(x)|h(k)> = e^{-ik\cdot x} \left(Z_h^{\mathrm{os}}\right)^{1/2} u_v(P)$$

Foldy–Wouthuysen

$$u(Mv+k) = \left[1 + \frac{\not{k}}{2M} + \mathcal{O}\left(\frac{k^2}{M^2}\right)\right] u_v(k)$$

Matching

Bare

$$Q_0(x) = e^{-iMv\cdot x}\left[z_0^{1/2}\left(1+\frac{i\not{D}_\perp}{2M}\right)h_{v0}(x)+\mathcal{O}\left(\frac{1}{M^2}\right)\right]$$

$$z_0 = \frac{Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)})}{Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)})} \qquad n_f = n_l + 1$$

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Renormalized

$$Q_0 = Z_Q^{1/2} Q(\mu) \qquad h_0 = Z_h^{1/2} h(\mu)$$

$$Q(\mu) = e^{-iMv\cdot x}\left[z(\mu)^{1/2}\left(1+\frac{i\not{D}_\perp}{2M}\right)h_v(\mu)+\mathcal{O}\left(\frac{1}{M^2}\right)\right]$$

$$z(\mu) = \frac{Z_h(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))}{Z_Q(\alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu))} z_0$$

Renormalized matching coefficient

$$z(\mu) = \frac{Z_h(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))\, Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)})}{Z_Q(\alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu))\, Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)})}$$

relates renormalized off-shell propagators $\Rightarrow$ finite at $\varepsilon \to 0$

- UV divergences cancel in Z_Q/Z_Q^{os}, Z_h/Z_h^{os}
- Z_Q, Z_h are IR finite
- IR divergences cancel in $Z_Q^{\text{os}}/Z_h^{\text{os}}$

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All light flavors are massless: $Z_h^{\text{os}} = 1$
no scale, UV and IR divergences mutually cancel

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RG equation $\gamma_i = d\log Z_i / d\log\mu$

$$\frac{d\log z(\mu)}{d\log\mu} = \gamma_h(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu)) - \gamma_Q(\alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu))$$

Renormalized matching coefficient

$$\log z(\mu) = \log Z_Q^{\mathrm{os}}(g_0^{(n_f)}, a_0^{(n_f)})$$
$$- \log Z_Q(\alpha_s^{(n_f)}(\mu), a^{(n_f)}(\mu)) + \log Z_h(\alpha_s^{(n_l)}(\mu), a^{(n_l)}(\mu))$$

- $g_0^{(n_f)} \to \alpha_s^{(n_f)}(\mu)$, $a_0^{(n_f)} \to a^{(n_f)}(\mu)$ — $\overline{\mathrm{MS}}$
- $\alpha_s^{(n_l)}(\mu) \to \alpha_s^{(n_f)}(\mu)$, $a^{(n_l)}(\mu) \to a^{(n_f)}(\mu)$ — decoupling
- $\mu = M$

Renormalized matching coefficient

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- α_s^1 up to ε^3
- α_s^2 up to ε^2
- α_s^3 up to ε^1
- α_s^4 up to ε^0

Renormalized matching coefficient

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- α_s^1 up to ε^3
- α_s^2 up to ε^2
- α_s^3 up to ε^1
- α_s^4 up to ε^0

Grozin (2010): up to α_s^3
(but positive powers of ε not presented)

Z_Q^{OS}

4 loops

- n_l^3 known for ages
- Lee, Marquard, Smirnov, Smirnov, Steinhauser (2013): n_l^2 analytically
- Marquard, Smirnov, Smirnov, Steinhauser (2018): all numerically
- Laporta (2020): C_F^4, $C_F^3 T_F n_h$, $C_F^2 (T_F n_h)^2$, $C_F (T_F n_h)^3$, $d_{FF} n_h$ numerically 1100 digits, analytically (for the last color structure, there are ε^0 terms of 6 non-planar light-by-light master integrals which are known up to 1100 digits)

$$d_{FF} = \frac{d_F^{abcd} d_F^{abcd}}{N_F} \qquad d_F^{abcd} = \operatorname{Tr} t_F^{(a} t_F^b t_F^c t_F^{d)}$$

Z_h

- $C_F(T_Fn_l)^3$ Broadhurst, Grozin (1995)
- $C_F^2(T_Fn_l)^2$ Grozin, Henn, Korchemsky, Marquard (2016)
- $C_FC_A(T_Fn_l)^2$ Marquard, Smirnov, Smirnov, Steinhauser (2018)
- $C_F^3T_Fn_l$ Grozin (2018)
- $d_{FF}n_l$ Grozin, Henn, Stahlhofen (2017)
- $C_F^2C_AT_Fn_l$, $C_FC_A^2T_Fn_l$ Brüser, Grozin, Henn, Stahlhofen (2019)
- $C_FC_A^3$, d_{FA} numerically Marquard, Smirnov, Smirnov, Steinhauser (2018)

Result

$z(\mu)$ must be finite $\Rightarrow$ all ε^{-n} terms in Z_Q^{os} analytically except $C_F C_A^3$, d_{FA} (C_F^4, $C_F^3 T_F n_h$, $C_F^2 (T_F n_h)^2$, $C_F (T_F n_h)^3$, $d_{FF} n_h$ independently obtained by Laporta)

Result

$z(\mu)$ must be finite $\Rightarrow$ all ε^{-n} terms in $Z_Q^{\rm os}$ analytically except $C_F C_A^3$, d_{FA} (C_F^4, $C_F^3 T_F n_h$, $C_F^2 (T_F n_h)^2$, $C_F (T_F n_h)^3$, $d_{FF} n_h$ independently obtained by Laporta)

$z(M)$

- C_F^4, $d_{FF} n_h$, $C_F^3 T_F n_h$, $C_F^2 (T_F n_h)^2$, $C_F (T_F n_h)^3$, $C_F^2 (T_F n_l)^2$, $C_F C_A (T_F n_l)^2$, $C_F T_F^3 n_h n_l^2$, $C_F (T_F n_l)^3$ analytically
- $C_F^3 C_A$, $C_F^2 C_A^2$, $C_F C_A^3$, d_{FA}, $C_F^2 C_A T_F n_h$, $C_F C_A^2 T_F n_h$, $C_F C_A (T_F n_h)^2$, $C_F^3 T_F n_l$, $C_F^2 C_A T_F n_l$, $C_F C_A^2 T_F n_l$, $C_F^2 T_F^2 n_h n_l$, $C_F C_A T_F^2 n_h n_l$, $C_F T_F^3 n_h^2 n_l$, $d_{FF} n_l$ numerically

Result

$$z(M) = 1 - \frac{4}{3}\frac{\alpha_s}{\pi} - \left(\frac{\alpha_s}{\pi}\right)^2 (17.45 - 1.33 n_l)$$
$$- \left(\frac{\alpha_s}{\pi}\right)^3 (262.42 - 0.78\xi - 35.81 n_l + 0.98 n_l^2)$$
$$- \left(\frac{\alpha_s}{\pi}\right)^4 \big[5137.72 - 15.67\xi + 1.07\xi^2$$
$$- (1030.82 - 0.71\xi) n_l + 60.30 n_l^2 - 1.00 n_l^3\big] + \mathcal{O}(\alpha_s^5)$$
$$\alpha_s = \alpha_s^{(n_f)}(M) \qquad \xi = 1 - a^{(n_f)}(M)$$

Result

$n_l = 4$, Landau gauge

$$z(M) = 1 - \frac{4}{3}\frac{\alpha_s}{\pi} - 12.12\left(\frac{\alpha_s}{\pi}\right)^2 - 134.11\left(\frac{\alpha_s}{\pi}\right)^3 - 1903.42\left(\frac{\alpha_s}{\pi}\right)^4$$

Result

$n_l = 4$, Landau gauge

$$z(M) = 1 - \frac{4}{3}\frac{\alpha_s}{\pi} - 12.12\left(\frac{\alpha_s}{\pi}\right)^2 - 134.11\left(\frac{\alpha_s}{\pi}\right)^3 - 1903.42\left(\frac{\alpha_s}{\pi}\right)^4$$

naive nonabelianization (large β_0 limit) Grozin (2010)

$$1 - \frac{4}{3}\frac{\alpha_s}{\pi} - 16.66\left(\frac{\alpha_s}{\pi}\right)^2 - 153.41\left(\frac{\alpha_s}{\pi}\right)^3 - 1953.40\left(\frac{\alpha_s}{\pi}\right)^4$$

IR renormalon at $u = \frac{1}{2}$

Effect of a lighter-flavor mass

Extra factor (μ independent)

$$z' = \frac{Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)}, m_0^{(n_f)}) Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)}, 0)}{Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)}, 0) Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)}, m_0^{(n_l)})}$$

$$Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)}, 0) = 1$$

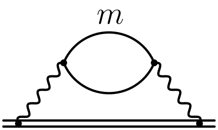
$$\log z' = \left(\log Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)}, m_0^{(n_f)}) - \log Z_Q^{\text{os}}(g_0^{(n_f)}, a_0^{(n_f)}, 0)\right)$$

$$- \log Z_h^{\text{os}}(g_0^{(n_l)}, a_0^{(n_l)}, m_0^{(n_l)})$$

Express via $\alpha_s^{(n_f)}(M)$, $a^{(n_f)}(M)$ and the on-shell m

$m \to 0$

HQET

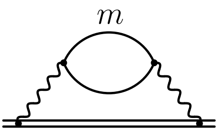


$$\log Z_h^{\mathrm{os}}(m) \sim g_0^4 m^{-4\varepsilon}$$

$$\log Z_h^{\mathrm{os}}(0) = 0$$

$m \to 0$

HQET



$$\log Z_h^{\mathrm{os}}(m) \sim g_0^4 m^{-4\varepsilon}$$

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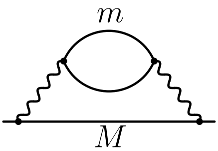
QCD



$$\log Z_Q^{\mathrm{os}}(0) \sim g_0^4 M^{-4\varepsilon}$$

$$\log Z_Q^{\mathrm{os}}(m)\Big|_{\mathrm{hard}} = \log Z_Q^{\mathrm{os}}(0)\left[1 + \mathcal{O}(x^2)\right]$$

$$\log Z_Q^{\mathrm{os}}(m)\Big|_{\mathrm{semisoft}} \sim g_0^4 M^{-2\varepsilon} m^{-2\varepsilon} x^2$$

$$\log Z_Q^{\mathrm{os}}(m)\Big|_{\mathrm{soft}} = \log Z_h^{\mathrm{os}}(m)\left[1 + \mathcal{O}(x)\right]$$

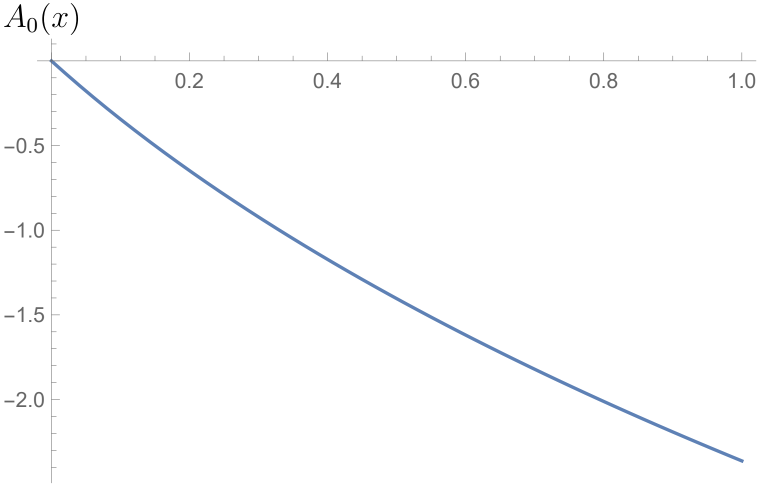
Result

3 loops

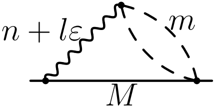
- $Z_Q^{\mathrm{os}}(m)$: Bekavac, Grozin, Seidel, Steinhauser (2007) truncated series in x
- $Z_h^{\mathrm{os}}(m)$: Grozin, Smirnov, Smirnov (2006)

$$
\begin{aligned}
z' = 1 &+ C_F T_F n_m \left(\frac{\alpha_s}{\pi}\right)^2 \left[A_0 + A_1 \varepsilon + \mathcal{O}(\varepsilon^2)\right] \\
&+ C_F T_F n_m \left(\frac{\alpha_s}{\pi}\right)^3 \Big[C_F A_F + C_A F_A \\
&\qquad + T_F n_l A_l + T_F n_m A_m + T_F n_h A_h + \mathcal{O}(\varepsilon)\Big]
\end{aligned}
$$

Result



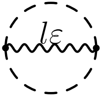
On-shell diagrams with 2 masses

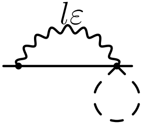


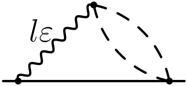
$$x = \frac{m}{M}$$

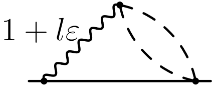
Davydychev, Grozin (1999)

- IBP

$j_1 =$ 

$j_2 =$ 

$j_3 =$ 

$j_4 =$ 

- Mellin–Barnes

$$\frac{x^{-2\varepsilon}}{2\pi i} \int_{-i\infty}^{+i\infty} ds\, x^{-2s} f(s)$$

Differential equations

- Kotikov (1991)

$$\partial_x j = M(x,\varepsilon) j$$

- Henn (2013) ε-form

$$j = T(x,\varepsilon) J \qquad \partial_x J = \varepsilon M(x) J \qquad M(x) = \sum_i \frac{M_i}{x - x_i}$$

Lee (2015) algorithm, Libra

Differential equations

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$$\partial_x j = M(x, \varepsilon) j$$

- Henn (2013) ε-form

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Lee (2015) algorithm, Libra

$$\frac{dJ}{dx} = \varepsilon \left(\frac{M_0}{x} + \frac{M_{+1}}{1-x} + \frac{M_{-1}}{1+x} \right) J$$

Differential equations

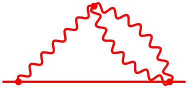
$$M_0 = \begin{pmatrix} -2(l+2) & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -(l+2) & l+2 \\ 1 & -1 & l+2 & -(l+2) \end{pmatrix}$$

$$M_{+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 2(l+2) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -2(l+2) & -2 \end{pmatrix}$$

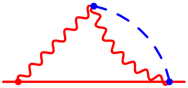
Boundary conditions $x \to 0$

Hard



$s = -n - \varepsilon$

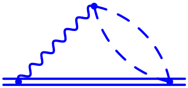
Semisoft



$s = -n$

double

Soft



$s = -n/2 + \varepsilon$

$x < 1$

$$
\begin{aligned}
J_3 = 2\Bigg\{ & -H_{1,0}(x) - H_{0,0}(x) - \frac{\pi^2}{3} \\
& + \Bigg[(l+2)\left(2H_{1,-1,0}(x) + H_{0,1,0}(x) + H_{0,-1,0}(x)\right) \\
& \quad - 2H_{1,1,0}(x) + 2(l+3)H_{0,0,0}(x) \\
& \quad + \frac{\pi^2}{6}\left(lH_1(x) + 2(l+3)H_0(x)\right) \\
& \quad - \frac{1}{2}(3l+2)\zeta_3 + (l+2)\pi^2 \log 2 \Bigg] \varepsilon + \mathcal{O}(\varepsilon^2) \Bigg\}
\end{aligned}
$$

$x < 1$

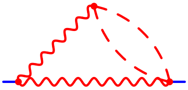
$$J_4 = 2\Bigg\{ H_{-1,0}(x) - H_{0,0}(x) + \frac{\pi^2}{6}$$
$$+ \Bigg[(l+2)\left(2H_{-1,1,0}(x) - H_{0,1,0}(x) - H_{1,-1,0}(x)\right)$$
$$- 2H_{-1,-1,0}(x) + 2(l+3)H_{0,0,0}(x)$$
$$+ \frac{\pi^2}{6}\left((5l+6)H_{-1}(x) - 2(2l+3)H_0(x)\right)$$
$$- \frac{1}{2}(3l+2)\zeta_3 - (l+2)\pi^2\log 2 \Bigg] \varepsilon + \mathcal{O}(\varepsilon^2) \Bigg\}$$

$x > 1$

$$\frac{dJ}{dx^{-1}} = \varepsilon \left(\frac{-M_0 + M_{+1} - M_{-1}}{x^{-1}} + \frac{M_{+1}}{1 - x^{-1}} + \frac{M_{-1}}{1 + x^{-1}} \right) J$$

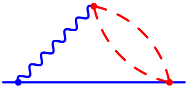
Boundary conditions $x \to \infty$

Hard



$s = n + \varepsilon$

Soft



$s = n$

QED and Bloch–Nordsieck EFT

- $z_0 = Z_\psi^{\text{os}}$ gauge invariant to all orders
- $\log Z_h = (3 - a^{(0)})\alpha^{(0)}/(4\pi\varepsilon)$ exactly ($\alpha^{(0)} = \alpha_{\text{os}}$)
- $\log Z_\psi = a^{(1)}\alpha^{(1)}/(4\pi\varepsilon) + \log Z_L$
- $a^{(1)}\alpha^{(1)} = a^{(0)}\alpha^{(0)}$
- gauge dependence cancels in $\log Z_h - \log Z_\psi$

$\log z = \log Z_\psi^{\text{os}} - \log Z_\psi + \log Z_h$ is gauge invariant

Result

$$z(M) = 1 - \frac{\alpha}{\pi} - 1.09991\left(\frac{\alpha}{\pi}\right)^2 + 4.40502\left(\frac{\alpha}{\pi}\right)^3 - 2.16215\left(\frac{\alpha}{\pi}\right)^4$$

$$\alpha = \alpha^{(1)}(M)$$

Conclusion

- All ε^{-n} terms in the 4-loop Z_Q^{os} are calculated analytically, except $C_F C_A^3$ and d_{FA} (the corresponding terms in γ_h are needed)
- $z(M)$ is calculated up to 4 loops, with corresponding lower-loop ε^n terms (most 4-loop terms only numerically)
- The mass correction z' is calculated up to 3 loops (most 3-loop terms as truncated series in x)
- The gauge-invariant QED $z(M)$ is calculated up to 4 loops (the last term "analytically")
- A class of on-shell integrals with 2 masses is investigated using differential equations in ε form

Possible application: heavy-quark propagators from the lattice

Electron field renormalization

Landau–Khalatnikov–Fradkin

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$S(x) = S_L(x)$$

Electron field renormalization

Landau–Khalatnikov–Fradkin

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x) e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))} \qquad \tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

Electron field renormalization

Landau–Khalatnikov–Fradkin

$$D^0_{\mu\nu}(k) = \frac{1}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \Delta(k) k_\mu k_\nu$$

$$S(x) = S_L(x) e^{-ie_0^2(\tilde{\Delta}(x) - \tilde{\Delta}(0))} \qquad \tilde{\Delta}(x) = \int \Delta(k) e^{-ikx} \frac{d^d k}{(2\pi)^d}$$

Massless electron

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&= \log Z_\psi + \sigma_r(x)
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